The Reorientation of a Human Being in Free Fall

by

Preston G. Smith
and
Thomas R. Kane

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ABSTRACT

The object of this work is to determine how a freely falling human being can change the orientation (attitude) of his body by moving parts of the body relative to each other. To this end, four models of the human body are studied in connection with the following two problems:
1) How does the orientation of each part of the body change when the relative motions of the parts proceed in certain specified ways?
2) How, if at all, can the parts be moved relative to each other so as to bring the model from a given initial state to a given final state?
In both problems it is assumed that the system is initially at rest so far as rotational motion is concerned. In the case of the first problem, results are presented for each of the four models considered; in three instances these results are in the form of plots showing possible re-orientations for a typical human being. A general method of solution is given for the second problem, and this method is applied to a model of the human body.

ACKNOWLEDGEMENT

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I. INTRODUCTION

1. Background

1.1 Manned exploration of space presupposes the ability to perform useful tasks while in a state of free fall ("weightlessness"). Since this state can be attained for only a few seconds at a time in experiments not involving orbital flight, little is known about the difficulties associated with performing specific tasks while falling freely. However, such knowledge is necessary in order to plan missions and train personnel, and it is thus natural to seek new means for obtaining the required information.

Man's mobility--his ability to change the position of his body--is of primary importance in this connection. This dissertation deals with a particular facet of this problem, namely with changes in orientation (attitude) of the body associated with relative motions of body parts.

Broadly speaking, this work deals with the application of principles of mechanics to human gross motor activity.† The following survey of past work in this field will not only provide useful background information, but should also indicate the motivation for the present work.

1.2 A number of books deal with human activity in a general way by reference to mechanics. In one by Fischer [1],‡ published in 1906, the body is considered to consist of n links. The mechanics of such a

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† "Gross motor activity" refers to the relative motions of the head, limbs, and major sections of the trunk, and will hereafter be called "motor activity" or just "activity."

‡ Numbers in brackets designate references listed at the end of the dissertation.

system is developed by starting with a three-link planar system and then extending the analysis to the three dimensional, n-link case. Equations of motion are obtained and, in the second part of the book, the author discusses applications of his theory to models of the human body. Some of the subjects covered here are: determination of masses, mass centers, and moments of inertia for the whole body and its parts, muscle forces and the resulting moments at the joints, and questions of equilibrium. However, no applications are given for the equations of motion. Thus, it must be concluded that Fischer does not really investigate the problem of motion of the human body.

A more modern book dealing with the mechanics of human activity was written by Williams, a physical therapist, and Lissner, an engineer [3]. In essence, it is a textbook on elementary statics as applied to the human body. Although it was written to present mechanics to the therapist, it is a useful reference on human anatomy for the engineer.

Several books, written primarily by physical educators, attempt to deal with the motion of the human body in a rather broad, but systematic way. Some authors, such as Broer [4], open with a rudimentary exposition of mechanics, the presumption being that an elementary understanding of force, acceleration, and angular momentum is both useful and, in some sense, adequate for an understanding of body motion. Others, for example, Cooper and Glassow [5], must feel that knowledge of mechanics is not prerequisite to an understanding of body motion, for their works contain a discussion of mechanics only as "additional material," if at all.

1.3 Several specific types of human motor activity have been studied, at least in part, by reference to principles of mechanics.

However, some of the investigators were not primarily mechanicians, and their use of mechanics was not always correct.

The activity which has probably been studied most extensively is locomotion, i.e., walking and running. While many authors in this field, Fenn [6,7,8] and Margaria and Cavagna [9], for example, make liberal use of some of the concepts of mechanics such as kinetic and potential energy, force, work, and the motions of pendulums, there appears to be little theoretical basis for their use of these concepts.

Perhaps the most complete work available on human motor activity was written by Amar [10] and originally published in French in 1914. Amar was interested in the efficiency of human activity in connection with industrial labor. Consequently, he considers not only the mechanics of body movement, but also questions of fatigue, diet, the conversion of chemical energy into mechanical energy, and the determination of the most efficient procedure for doing a given task. The book commences with a section on "the general principles of mechanics" which includes elementary statics and dynamics and the structural properties of the materials of which the body is constructed.

The mechanics† of some athletic events have been studied by a few investigators, usually coaches who have had little or no formal training in analytical mechanics. When an analysis based on analytical mechanics is attempted at all, it usually deals with an implement used, e.g., the bending of a vaulting pole or the aerodynamics of a javelin, rather than the athlete himself. Cureton [11-15], in 1935, wrote a series of articles

† "Mechanics" in this context is often intended in its more general sense: "an analysis of the act in terms of the dimensional, force and time relationships" (from Cureton [11]).

on the mechanics of track running, the track racing start, the shot put, the high jump, and the broad jump. These articles generally consist of vague statements about forces of action and reaction, power, and various classes of levers. Cureton also studied the mechanics of swimming [16], and Counsilman [17] determined the resistive and propulsive forces of swimmers experimentally.

A book on the mechanics of the pole vault was written by Richard Ganslen [18], a champion vaulter who is considered "the world's greatest authority" [19] on the subject. Suffice it to say that this book concludes with a list of the "Laws of Mechanics," two of which are "To accelerate angular velocity in swinging exercises, lengthen the radius of gyration on the downswing and shorten the radius on the up-swing and "Placing of one arm in an unbalanced position creates a gyroscopic effect."

In a volume in which he introduces those aspects of mechanics which are particularly useful for the analysis of athletic events, Dyson [20] was probably more successful than any other author in describing track and field events in terms of the principles of mechanics. However, since mechanics was employed in a very elementary, qualitative manner, one may well ask whether the descriptions were devised to agree with observations or whether they are actually valid statements of the essential physical relationships.

1.4 Anthropometry is the science of measuring the human body and its parts. Measurements such as those of length, mass, location of the mass center, moment of inertia, or kinematical constraint either use the principles of mechanics for their determination or are of use to the

engineer when studying the human body.

The work of Braune and Fischer [21] on the location of the mass center of the human body was published in 1889 and stands as a landmark in anthropometry. The measurements were made on four frozen cadavers. After determining the location of the mass center of individual parts of the body, a cartesian coordinate system was used to locate the various parts in relation to each other. This allowed the description of various body positions and the calculation of the location of the mass center of the whole body for these positions. In this way, Braune and Fisher determined the location of the mass center of the human body for several situations of military interest.

The work of Braune and Fischer was extended by Dempster [22], who in 1955 reported on an additional eight cadavers. However, Dempster's primary contribution was the determination of the anatomical constraints imposed at the joints of the human body. His results were presented graphically, and, in addition, he gave detailed instructions for the construction of a kinematically correct manikin.

The nation's space program has created a need for information of the type given by Braune and Fischer and Dempster. Most of this work is currently being done at the Aerospace Medical Research Laboratories, Wright-Patterson Air Force Base, Ohio. In one study, reported in 1963, Santschi, Du Bois, and Omoto [23] measured the location of the mass center as well as the moments of inertia for the mass center of the living human body in eight selected positions. A similar study [24] was made for a seated subject wearing a space suit.

Hanavan [25,26], in 1964, presented a mathematical model for the

determination of inertial properties of the human body. The model consisted of fifteen rigid bodies, and the body position was defined by giving the relative orientations of the fifteen component bodies. Numerical results obtained by using the model were compared with those reported for the experimental studies mentioned in the previous paragraph. Hanavan's results were presented as a tabulation of the mass center locations and moments of inertia for 31 body positions.

1.5 One of the most intriguing and widely discussed problems in animal motion is that of a cat which, when falling, is able to turn over in mid-air so as to land on its feet. Marey [27] photographed a falling cat in 1894, and when his photographs were shown to the French Académie des Sciences, they evoked a discussion on whether an initial angular velocity was necessary in order to perform the righting maneuver. Guyou [28], a physicist, gave an explanation of the phenomenon for the case when the angular momentum of the entire cat was at all times zero.

Guyou considered the cat as being composed of two rigid bodies which were able to rotate relative to each other at the waist. Each body had a moment of inertia about the axis of rotation of value I when the legs of that half of the body were extended and of value i when they were pulled in toward the trunk. The respective angles of rotation of the bodies were ω and Ω, and these quantities were related by the equation

$$\frac{\Omega}{\omega} = \frac{I}{i} \qquad (1.5.1)$$

According to Guyou, the maneuver was initiated by extending the rear legs and retracting the front ones. A twist at the waist produced a

large rotation Ω_1 of the front part and a smaller counterrotation ω_1 of the rear part. Then the front legs were extended, the rear ones retracted, and the cat twisted back into its original position at the waist, producing rotations ω_2 and Ω_2 of the front and rear parts; the total change in orientation was thus $\Omega_1 - \omega_2$. This explanation, based on an elementary understanding of mechanics, is the one most often given today; see, for example, [29] and [30].

The next observer to study the falling cat in detail was Magnus, a Dutch physiologist [31]. In 1922, he published a series of photographs and a more detailed explanation of the righting maneuver. Magnus was concerned with reflexes rather than with the mechanics of the problem, and he divided the maneuver into three distinct phases: rotation of the head, rotation of the fore-body, and rotation of the rear part.

Rademaker and Ter Braak [32], after examining their own photographs of a falling cat, proposed a new explanation in 1935. According to these authors the cat bends at the waist, and the front and rear parts then rotate together, bringing the whole cat over at once. Their model consisted of two identical circular cylinders connected at the waist. The symmetry axes of the cylinders defined a plane. They showed (see Appendix 1), by angular momentum arguments, that if the cylinders rotate relative to the plane with equal rates, then the plane itself will also rotate, the result being a turning-over of the whole model.

In 1955, McDonald [33,34,35] published results of an extensive study in which he had photographed falling cats with a high-speed (1500 frames/sec.) motion picture camera. He described their motion generally in terms of Guyou's explanation, but added many details seen in his observations. In particular, McDonald found no evidence for either Magnus'

observation that the head rotates before the rest of the body, or for the Rademaker-Ter Braak explanation which requires the front and rear parts to rotate together. Dr. McDonald also made some inertia measurements on parts of a cat and, using (1.5.1), calculated that a rotation of 180 degrees of the fore-part could be attained by a counterrotation of six degrees of the hind-part. This was confirmed by measurements taken from his photographs. After photographing a tailless cat, he concluded that the tail was not necessary for the maneuver, but it was sometimes useful for fine adjustments or for stopping the rotation once the desired orientation had been achieved. In addition, McDonald studied the role of the eyes and the vestibular organ as orientation and motion sensors.

1.6 After studying attitude motions of a freely falling cat, McDonald did the same for a man [36]. In one experiment, the subjects were asked to jump from a one meter springboard and to attempt to perform a rotation about a vertical (longitudinal) axis before reaching the water. To ensure that the angular motion was not obtained from the push-off, they were told the direction in which to turn after they left the board. It was found that a highly trained diver could make a full rotation in either direction, but an untrained person could turn only 90 degrees at most.

The subjects were also asked to try the "cat drop" maneuver. They hung from the underside of a diving board by their hands and feet. On command, they let go and attempted to turn over, landing face down with their hands and feet entering the water first. The trained diver was successful at the "cat drop"; the others were not.

Whitsett [37] considered the motion of a space worker (consisting of

a single "weightless" body) when subjected to the action of forces resulting from thrust misalignment or from the application of a torque to a fixed handle. The space worker's propulsion system produces a force which is applied near the worker's mass center. Thrust misalignment, due to the force not passing exactly through the mass center, causes unwanted rotational motion. Whitsett studied this motion, but he considered only misalignments which produce a torque about a principal axis of the body. In practice, thrust misalignments will not be aligned with the principal axes. Hence, Whitsett considered only a special case of this problem.

Tieber and Lindemuth [38] studied the motion of an astronaut, considered as a single rigid body, when he is using a Modular Maneuvering Unit. This device consists of twelve gas jets, a fuel supply, and controls which enable the astronaut to maneuver himself while outside his spacecraft.

Several workers have studied the attitude motion of an astronaut in free fall, taking into account motions of parts of the body relative to one another. McCrank and Seger [39] considered the human body to be composed of nine connected rigid bodies (a trunk-head and four limbs of two parts each). They developed a procedure for writing equations of motion for an n-body system in free fall and applied the procedure to their nine body system. The resulting differential equations of motion were quite complex, and no general conclusions were drawn. However, some numerical integrations for specific cases were presented.

The problem of the reorientation of a human being in free fall due to movement of the limbs was studied by Kulwicki, Schlei, and Vergamini [40] in the U.S.A. and by Stephantsov, Yeremin, and Alekperov [41] in

the U.S.S.R. The theoretical basis for both reports was the same; namely, the equation $T = I\alpha$, where the angular acceleration α is assumed to be the second derivative of an angle θ, was integrated twice to give a relationship essentially the same as (1.5.1). This relationship was referred to by both sets of authors in order to "prove" that each of several specific maneuvers would result in a reorientation. In some cases, it was not clear how the "theory" related to the proposed maneuvers (e.g., the pinwheel, p. 9 of Kulwicki). In addition, the analysis was based on some assumptions which were not discussed by the authors and which were violated to some extent in all of the proposed maneuvers. For instance, $T = I\alpha$ and the expression of α as the second derivative of θ are valid for planar motion only (see the bend and twist, Kulwicki, p. 6, for a violation of the planar motion hypothesis). More subtle is the fact that $T = I\alpha$ is based on the angular momentum principle and is valid only when the axis of rotation of each of the bodies is either inertially fixed (not generally true for freely falling bodies) or passes through the mass center of the system (true only in special cases of symmetry). For example, the statement "If the subject exerts muscular force and turns the body part A through an angle a, in unit time, then the other part of the body, according to the Third Law of Mechanics, turns in the opposite direction...." (Stepantsov, p. 5) is not always true (refer to Section 3.3 of the present work for an example). The third failing of this analysis is that it takes no account of the reorientations caused by certain intermediate parts of the maneuver, during which the moments of inertia of the two parts are changed (e.g., steps 3 and 5 of the bend and twist, p. 6 of Kulwicki).

The shortcomings of the analysis cast varying amounts of doubt on the conclusions reached for each of the proposed maneuvers. Two methods are available for testing these conclusions: experimental verification or a more thorough analysis. Stepantsov et al. tested their conclusions experimentally on a low friction platform and on a trampoline and suggested that certain maneuvers also be tried in an aircraft flying in a Kepler orbit. Kulwicki et al. did not verify their conclusions, but also suggested verification in an aircraft. (The present author has found by personal correspondence that a test program involving all nine of the proposed maneuvers was carried out, but that no report was published.) However, experimental verification has its disadvantages: a frictionless platform, or Zhukovskiy bench, as it is called, does not by any means furnish a free fall condition; the trampoline allows very little time for the maneuver and the initial conditions are uncertain; and aircraft flights are both expensive and time consuming. It is one of the purposes of this dissertation to perform more thorough analysis and thus to reduce the need for experimental verification.

2. Problem Statement

2.1 For purposes of the present analysis, the human body is regarded as a collection of rigid bodies which are connected to each other at joints that allow some amount of relative rotational motion. The bodies are known collectively as the system or model. One body of the system is designated the main body, and its orientation with respect to an inertial reference frame is specified in terms of three quantities, called external variables. The orientations of the bodies with respect

to each other depend on certain angles, called internal variables. The state of the system is known when both the internal and external variables are known. The system is said to be in "free fall" when it is subject to the action of no external forces other than a gravitational force applied at the mass center. During free fall, rotational motions of the system are unaffected by translational motions of the mass center. Consequently, the dynamical relationship between internal and external variables depends solely on the initial angular velocities of the bodies.

2.2 The relationship between internal and external variables may be examined by reference to two problems. In one of these, hereafter called the First Problem, the internal variables are specified as functions of time, and the equations of motion are solved as an initial value problem to determine the time dependence of the external variables. The effect of initial conditions is eliminated by always starting with all bodies at rest so far as rotational motion is concerned. The First Problem is examined in Chapter II, where solutions are obtained for several models.

Chapter III is devoted to the Second Problem, which consists of determining a time history of the internal variables such that the system moves from a specified initial state of rest to a specified final state. This problem does not necessarily possess a solution; i.e., some final states may be unattainable. On the other hand, there may exist more than one solution.

II. THE FIRST PROBLEM

3. Two Bodies in Planar Motion

3.1 Consider a system, S, composed of two bodies, B_0 and B_1, which are constrained to move in a plane. B_0 and B_1 are connected by a hinge at 0, the axis of the hinge being normal to the plane of motion. The masses of B_0 and B_1 are m_0 and m_1, respectively, and the moments of inertia of B_0 and B_1 about axes passing through the respective mass centers, C_0 and C_1, and normal to the plane of motion are I_0 and I_1. Finally, the internal variable φ and the external variable ξ are defined as indicated in Fig. 3.1.

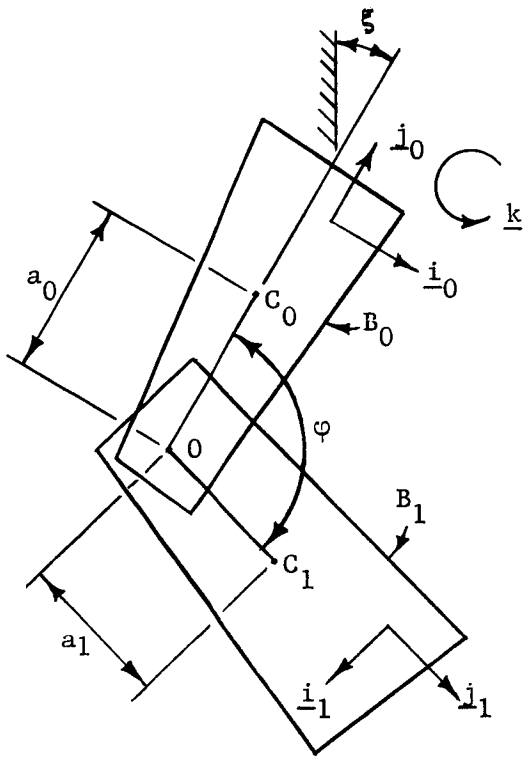

Fig. 3.1

3.2 Equations of motion, in this section as well as later, are obtained by use of the angular momentum principle. As was pointed out in the introduction, "free fall" will be taken to mean that the resultant of all forces acting on the system passes through the mass center of the system. It then follows from the angular momentum principle that the angular momentum of the system with respect to the mass center remains constant in an inertial reference frame.

For two reasons, it is assumed that the system is initially at rest: firstly, this simplifies the equations of motion because the external variables need not enter the equations explicitly (cf. (5.2.9)-(5.2.11)); secondly, since the external variables depend on initial conditions as well as on the internal variables, the relationship between the two sets of variables is simplified when the motion starts from rest.

The angular momentum, $\underline{A}^{S/C^*}$, of S with respect to C^*, the mass center of S, is composed of four terms:

$$\underline{A}^{S/C^*} = \underline{A}^{B_0/C_0} + \underline{A}^{C_0/C^*} + \underline{A}^{B_1/C_1} + \underline{A}^{C_1/C^*} \qquad (3.2.1)$$

where $\underline{A}^{B_0/C_0}$ is the angular momentum of B_0 with respect to C_0; $\underline{A}^{C_0/C^*}$ is the angular momentum with respect to C^* of a particle of mass m_0 situated at C_0; and $\underline{A}^{B_1/C_1}$ and $\underline{A}^{C_1/C^*}$ are defined likewise. The first and third terms of (3.2.1) are given by†

$$\underline{A}^{B_0/C_0} = -\dot{\xi}\, I_0\, \underline{k} \quad , \quad \underline{A}^{B_1/C_1} = -(\dot{\xi} + \dot{\varphi})\, I_1\, \underline{k} \qquad (3.2.2)$$

whereas the second and fourth terms may be written

---

† A dot above a scalar quantity denotes differentiation with respect to time. The symbol d/dt, when applied to a vector, signifies differentiation in an inertial reference frame.

$$\underline{A}^{C_0/C^*} = \underline{r}_0 \times m_0 \frac{d}{dt} \underline{r}_0 \quad , \quad \underline{A}^{C_1/C^*} = \underline{r}_1 \times m_1 \frac{d}{dt} \underline{r}_1 \qquad (3.2.3)$$

where $\underline{r}_0$ and $\underline{r}_1$ are the position vectors of C_0 and C_1 relative to C^*:

$$\underline{r}_0 = \frac{m_1}{m_0 + m_1} (a_0 \underline{j}_0 - a_1 \underline{j}_1) \quad , \quad \underline{r}_1 = \frac{m_0}{m_0 + m_1} (a_1 \underline{j}_1 - a_0 \underline{j}_0) \qquad (3.2.4)$$

All vector quantities in (3.2.1) through (3.2.4) may be expressed in terms of unit vectors fixed in the main body, B_0, by use of the substitutions

$$\underline{i}_1 = \cos \varphi \, \underline{i}_0 - \sin \varphi \, \underline{j}_0 \quad , \quad \underline{j}_1 = \sin \varphi \, \underline{i}_0 + \cos \varphi \, \underline{j}_0 \qquad (3.2.5)$$

Use of (3.2.2) through (3.2.5) in (3.2.1) results in the following expression for the angular momentum of S relative to C^*:

$$\underline{A}^{S/C^*} = -\left\{\left[I_0 + I_1 + \frac{m_0 m_1}{m_0 + m_1} (a_0^2 + a_1^2 - 2a_0 a_1 \cos \varphi)\right] \dot{\xi} + \left[I_1 + \frac{m_0 m_1}{m_0 + m_1} (a_1^2 - a_0 a_1 \cos \varphi)\right] \dot{\varphi}\right\} \underline{k} \qquad (3.2.6)$$

Equating this expression to zero yields the differential equation of motion†

$$[(m_0 + m_1)(I_0 + I_1) + m_0 m_1 (a_0^2 + a_1^2 - 2 a_0 a_1 \cos \varphi)] \xi' = -[(m_0 + m_1) I_1 + m_0 m_1 (a_1^2 - a_0 a_1 \cos \varphi)] \varphi' \qquad (3.2.7)$$

---

† Time t has been removed from (3.2.7) by introducing a new independent variable $\tau = t/T$ where T is the time required for the complete maneuver; primes denote differentiation with respect to τ. This substitution is made here, as well as later, in order to eliminate t from solutions of equations of motion and to standardize the duration of the maneuver.

which may be integrated over the duration of the maneuver, $0 \leq \tau \leq 1$, to give

$$\xi(1) - \xi_0 = - \int_{\varphi_0}^{\varphi(1)} \frac{a + \cos \varphi}{b + 2 \cos \varphi} \, d\varphi \qquad (3.2.8)$$

where

$$a = - \frac{(m_0 + m_1) I_1}{m_0 m_1 a_0 a_1} - \frac{a_1}{a_0}$$

$$b = - \frac{(m_0 + m_1)(I_0 + I_1)}{m_0 m_1 a_0 a_1} - \frac{a_0}{a_1} - \frac{a_1}{a_0} \qquad (3.2.9)$$

and

$$\xi_0 = \xi(0) \quad , \quad \varphi_0 = \varphi(0) \qquad (3.2.10)$$

As $b < -2$, the integration indicated in (3.2.8) yields

$$\xi(1) - \xi_0 = \left[- \frac{\varphi}{2} + \frac{b - 2a}{\sqrt{b^2 - 4}} \tan^{-1} \left(\sqrt{\frac{b^2 - 4}{b + 2}} \tan \frac{\varphi}{2} \right) \right]_{\varphi_0}^{\varphi(1)} \qquad (3.2.11)$$

3.3 When B_0 and B_1 are taken to be uniform thin rods joined at the ends, the quantities a and b may be expressed solely in terms of the ratio of the lengths a_0 and a_1:

$$a = - \frac{4}{3} \frac{a_1}{a_0} - \frac{1}{3} \left(\frac{a_1}{a_0} \right)^2$$

$$b = - \frac{1}{3} \left(\frac{a_1}{a_0} \right)^{-2} - \frac{4}{3} \left(\frac{a_1}{a_0} \right)^{-1} - \frac{4}{3} \frac{a_1}{a_0} - \frac{1}{3} \left(\frac{a_1}{a_0} \right)^2 \qquad (3.3.1)$$

The relationship between the internal variable φ and the external variable ξ may be explored by starting with the initial conditions

$\varphi_0 = \xi_0 = 0$ and plotting the final values $\xi(1)$ vs. $\varphi(1)$ for various ratios of the lengths. Such a plot is given in Fig. 3.3a. One characteristic of this graph is of particular interest: for some length ratios, e.g., $\frac{a_1}{a_0} = \frac{3}{10}$, the angles φ and ξ, when starting from zero, both increase positively over some interval. In other words, the system maintains a zero-valued angular momentum, in this instance, when both rods are rotating in the same sense. Fig. 3.3b illustrates the motion of S relative to C* when $\frac{a_1}{a_0} = \frac{3}{10}$ by showing S in initial, intermediate, and final positions.

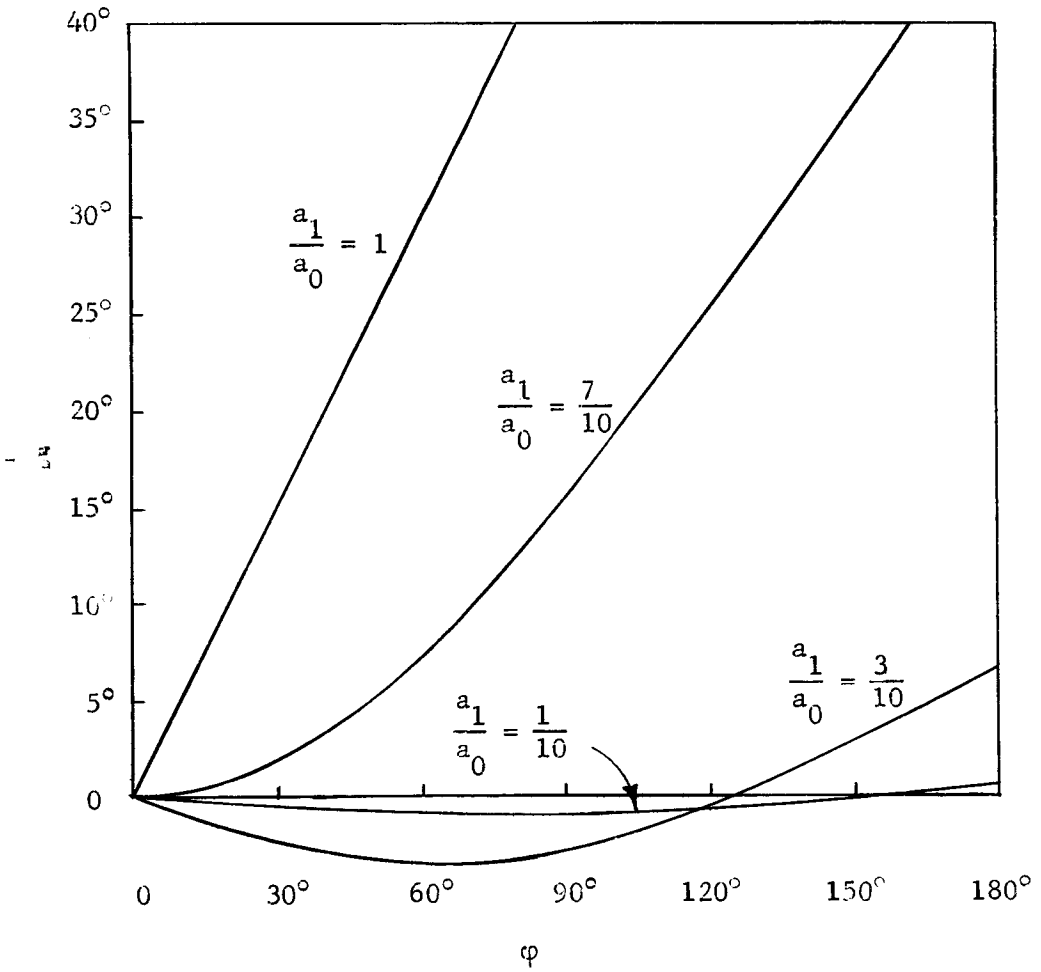

Fig. 3.3a

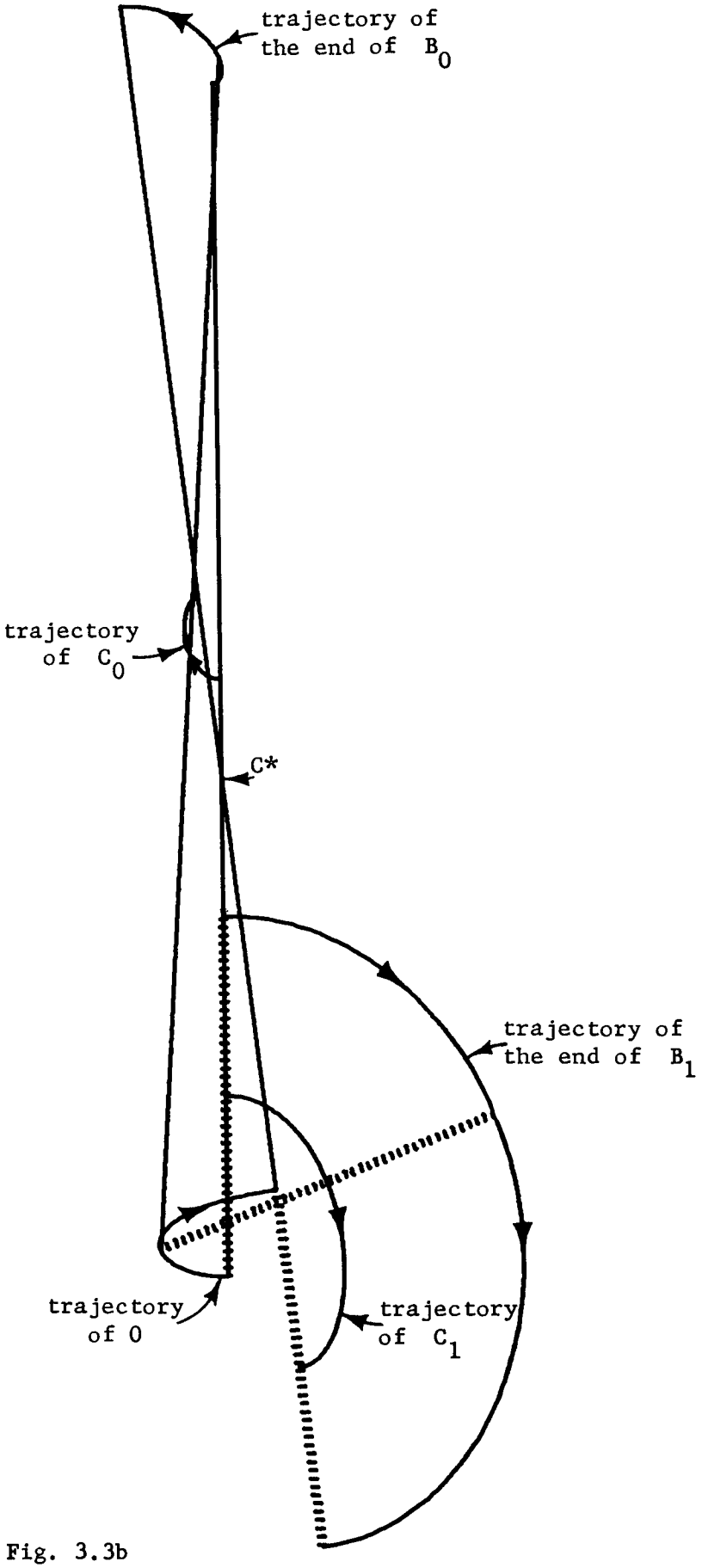
trajectory of
the end of B_0
trajectory
of C_0
C*
trajectory of
the end of B_1
trajectory
of 0
trajectory
of C_1

Fig. 3.3b

4. Three Bodies in Planar Motion

4.1 The system, S, is an extension of the one examined in the last section:† three bodies are connected by two hinges and are constrained to move in a plane which is perpendicular to the hinge axes. The main body, B_0, is connected to each of the other two, B_1 and B_2, by hinges located at O_1 and O_2, respectively. The bodies have masses m_0, m_1, and m_2 (whose sum is m, the mass of S) and moments of inertia I_0, I_1, and I_2 about axes normal to the plane of motion and passing through the respective mass centers C_0, C_1, and C_2. The two internal variables are φ and ψ, the external one being ξ as shown in Fig 4.1.

4.2 The equation of motion is obtained, as before, by computing the angular momentum of S relative to C*, the mass center of S, and then setting the result equal to zero.

The angular momentum may be expressed as the sum of six terms:

$$\underline{A}^{S/C^*} = \underline{A}^{B_0/C_0} + \underline{A}^{B_1/C_1} + \underline{A}^{B_2/C_2} + \underline{A}^{C_0/C^*} + \underline{A}^{C_1/C^*} + \underline{A}^{C_2/C^*} \qquad (4.2.1)$$

Of these, the first three deal with motions of the three bodies relative to their respective mass centers, and the last three are associated with motions of the mass centers. The first term of (4.2.1) is, for example,

$$\underline{A}^{B_0/C_0} = - \dot{\xi} I_0 \underline{k} \qquad (4.2.2)$$

and the fourth term is

$$\underline{A}^{C_0/C^*} = \underline{r}_0 \times m_0 \frac{d}{dt} \underline{r}_0 \qquad (4.2.3)$$

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† The symbol S now has a meaning different from that in Section 3; all symbols are defined anew in each section.

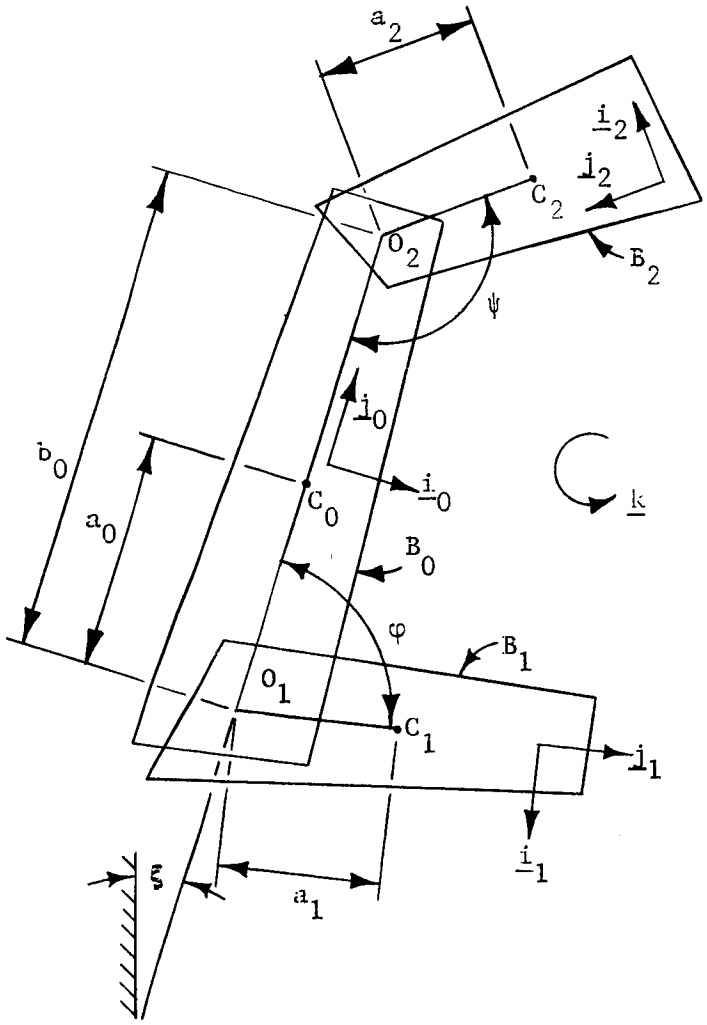

Fig. 4.1

where $\underline{r}_0$ is the position vector of C_0 relative to C^*.

By letting $\underline{r}$ denote the position vector of O_1 relative to C^*, $\underline{r}_0$ may be written as $\underline{r} + a_0\, \underline{j}_0$, and similarly, the positions of C_1 and C_2 may be expressed as the sum of $\underline{r}$ and vectors fixed in the bodies. The vector $\underline{r}$ may be computed by expressing the position of C^* relative to O_1 in two different ways and equating them:

$$-\underline{r} = \frac{1}{m}\left[m_0\, a_0\, \underline{j}_0 + m_1\, a_1\, \underline{j}_1 + m_2\, (b_0\, \underline{j}_0 - a_2\, \underline{j}_2)\right] \qquad (4.2.4)$$

The angular momentum of S relative to C* may be expressed as

$$\underline{A}^{S/C^*} = \frac{1}{m^2} (h_1 \dot{\varphi} + h_2 \dot{\psi} - h \dot{\xi}) \underline{k} \qquad (4.2.5)$$

where

$$h_1 = - m_1 m_2 m a_1 a_2 \cos(\varphi + \psi) + m_1 m a_1 (m_0 a_0 + m_2 b_0) \cos\varphi$$
$$- m^2 I_1 - m_1 m(m_0 + m_2) a_1^2$$

$$h_2 = m_1 m_2 m a_1 a_2 \cos(\varphi + \psi) - m_2 m a_2[(m_0 + m_1) b_0$$
$$- m_0 a_0] \cos\psi + m^2 I_2 + m_2 m (m_0 + m_1) a_2^2$$

$$h = 2 m_1 m_2 m a_1 a_2 \cos(\varphi + \psi) - 2 m_1 m a_1 (m_0 a_0 + m_2 b_0) \cos\varphi$$
$$- 2 m_2 m a_2 [(m_0 + m_1) b_0 - m_0 a_0] \cos\psi + m^2(I_0 + I_1 + I_2)$$
$$+ 2 m_1 m_2 m a_1 a_2 + m_0 (m_1 a_1 + m_2 a_2)^2 + m_0[(m_1 + m_2) a_0$$
$$- m_2 b_0]^2 + m_1[(m_0 + m_2) a_1 - m_2 a_2]^2 + m_1(m_0 a_0 + m_2 b_0)^2$$
$$+ m_2[(m_0 + m_1) a_2 - m_1 a_1]^2 + m_2[(m_0 + m_1)b_0 - m_0 a_0]^2 \qquad (4.2.6)$$

From (4.2.5), the equation of motion is

$$f_1 \varphi' + f_2 \psi' - \xi' = 0 \qquad (4.2.7)$$

where f_1 and f_2 are defined by reference to (4.2.6):

$$f_1 = f_1(\varphi, \psi) = \frac{h_1}{h} \quad , \quad f_2 = f_2(\varphi, \psi) = \frac{h_2}{h} \qquad (4.2.8)$$

Only eight of the ten quantities (exclusive of φ and ψ) which enter into (4.2.6) are necessary to determine the motion. Consequently, eight dimensionless parameters are introduced:

$$H_0 = \frac{I_0}{m\, b_0^2} \quad , \quad H_1 = \frac{I_1}{m\, b_0^2} \quad , \quad H_2 = \frac{I_2}{m\, b_0^2}$$

$$p_0 = \frac{a_0}{b_0} \quad , \quad p_1 = \frac{a_1}{b_0} \quad , \quad p_2 = \frac{a_2}{b_0} \quad , \quad n_1 = \frac{m_1}{m} \quad , \quad n_2 = \frac{m_2}{m} \qquad (4.2.9)$$

4.3 A comparison of (3.2.7) and (4.2.7) reveals a fundamental difference. In view of (3.2.8), the solution of (3.2.7) may be expressed as

$$\xi(1) - \xi_0 = \int_{\varphi_0}^{\varphi(1)} f(\varphi)\, d\varphi \qquad (4.3.1)$$

Likewise, (4.2.7) implies that

$$\xi(1) - \xi_0 = \int_{\varphi_0}^{\varphi(1)} f_1(\varphi, \psi)\, d\varphi + \int_{\psi_0}^{\psi(1)} f_2(\varphi, \psi)\, d\psi \qquad (4.3.2)$$

Now, the right-hand side of (4.3.1) is expressed as a quadrature, and it follows that the value of $\xi(1) - \xi_0$ depends only on φ_0 and $\varphi(1)$. In particular, if $\varphi(1)$ equals φ_0, then ξ_0 and $\xi(1)$ must be equal, which means that if the internal variable returns to its original value, then the external variable must do likewise. In contrast, the right-hand side of (4.3.2) has not been reduced to quadratures, each integral containing a variable in addition to the integration variable. Since this additional variable may take on different values throughout the interval of integration, the value of the first integral in (4.3.2), for example, depends not only on φ_0 and $\varphi(1)$, but also on the time histories of both φ and ψ. Thus, the values of the integrals are not necessarily zero when the limits of integration are equal, and there is now no guarantee that the external variables return to their original values whenever all of the internal variables recover their initial values.

4.4 The nature of the solutions of (4.2.7) may be studied by integrating the equation numerically on a digital computer for several different time histories of φ and ψ. Specifically, a "standard" time history containing several parameters may be used, the parameters being varied in order to determine their effect on ξ.

The standard time history consists of portions in which the variable remains constant and portions in which it changes according to the function

$$y(t) = y_a + (y_b - y_a)\left(\frac{t}{T} - \frac{1}{2\pi}\sin\frac{2\pi t}{T}\right) \qquad (4.4.1)$$

where y_a and y_b are the values of y at the beginning and end of the transition, the duration of which is T. As shown in Fig. 4.5a, the standard time history has the following general form: it begins with a certain initial value, changes to an intermediate value, and returns to the initial value. Seven parameters determine the exact shape of the curve, these being the initial and intermediate values and the durations of the two transitional and three constant portions of the history.

Two properties of (4.4.1) are noteworthy: the slope is zero at both ends of the transition (ensuring that the motion starts from rest as stipulated); and the second derivative (and thus the inertia force) is continuous throughout the history.

4.5 Numerical integration of (4.2.7) for a specific system reveals properties of the solution which give some general insight into the behavior of systems of connected rigid bodies in free fall. Supposing S to consist of three identical uniform thin rods connected at the ends, the dimensionless parameters (4.2.9) assume the values

$$H_0 = \frac{1}{36} \quad , \quad H_1 = \frac{1}{36} \quad , \quad H_2 = \frac{1}{36}$$

$$p_0 = \frac{1}{2} \quad , \quad p_1 = \frac{1}{2} \quad , \quad p_2 = \frac{1}{2}$$

$$n_1 = \frac{1}{3} \quad , \quad n_2 = \frac{1}{3}$$

If the internal variables are now restricted to values between 0 and 180 degrees, one can pose the following problem: with the stipulation that the final values of the internal variables be the same as the initial values, how must these variables be chosen so as to yield the largest possible reorientation of the system? This problem is solved here by first restricting the internal variables to functions which can be represented by the standard time history and then investigating the more general case.

If φ and ψ are prescribed by means of the standard time history, then the reorientation is a function of the parameters used in the history, and the problem at hand is that of maximizing a function of several variables. Because of the large number of parameters available in the standard time history, it is difficult to determine the maximum reorientation with absolute certainty; but by integrating (4.2.7) a sufficient number of times, each time using sets of parameters chosen on the basis of the results of prior integrations, a maximum reorientation may be obtained with a reasonable amount of confidence.

Fig. 4.5a shows time histories obtained by applying this procedure to the problem under consideration. It may be seen that the internal variables return to their original values, as required, whereas the external variable changes by about 104 degrees. These values, which appear to be optimal, are seen to be obtained by making full use of the

allowable range of the internal variables, and it is worth noting both that the range over which the external variable varies is larger than the difference between the initial and final values of this variable, and that the maximum reorientation is obtained by changing the internal variables one at a time.

Figs. 4.5b and 4.5c illustrate the effects of the two basic types of modification that can be made in the standard time history parameters. As may be seen from Fig. 4.5b, reductions in the range of φ and ψ generally result in a large reduction in the reorientation; specifically, if φ and ψ, instead of closing to zero, only close to right angles, the reorientation obtained is only 7.5 degrees. On the other hand, modifications in the relative timing of φ and ψ have relatively little effect on the reorientation; this is illustrated in Fig. 4.5c in which the transition in ψ starts after only 20% (100% in Fig. 4.5a) of the transition in φ has occurred and in which the reorientation is 74 degrees.

If the internal variables are not restricted to functions which can be generated by use of the standard time history, it becomes more difficult to determine the maximum reorientation, and it may be shown that no such maximum exists unless φ and ψ are restricted in some way. In fact, consider two periodic functions, $y_1(t)$ and $y_2(t)$, of a unit period, and assume that these satisfy the restrictions on φ and ψ, respectively, and yield a reorientation of amount Ξ of S; then $y_1(nt)$ and $y_2(nt)$, where n is any positive integer, give rise to a reorientation of amount $n\Xi$ which has no maximum.

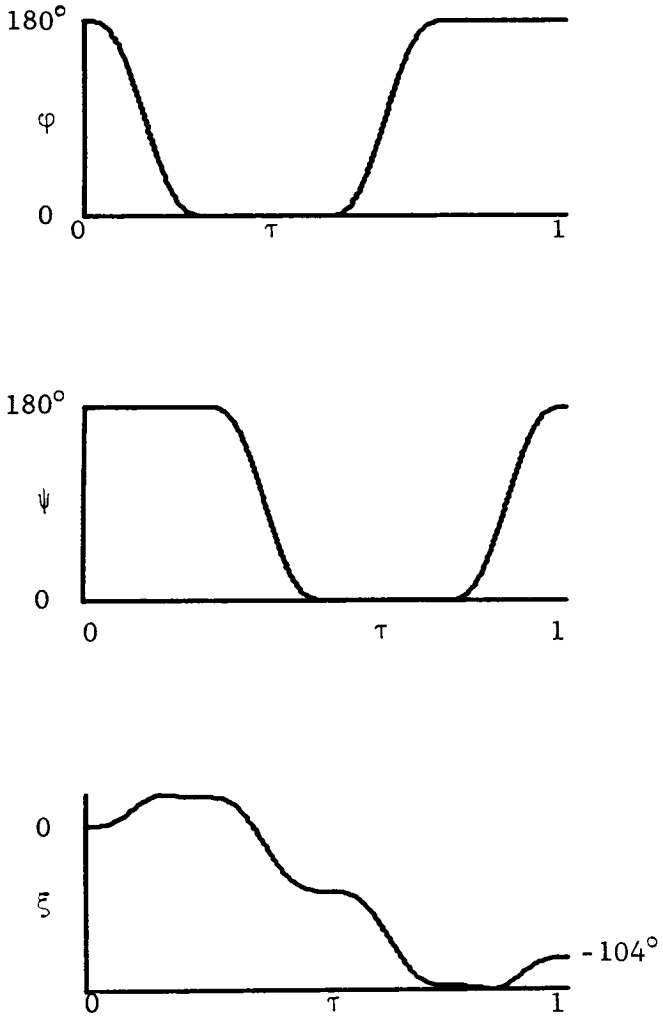

Fig. 4.5a

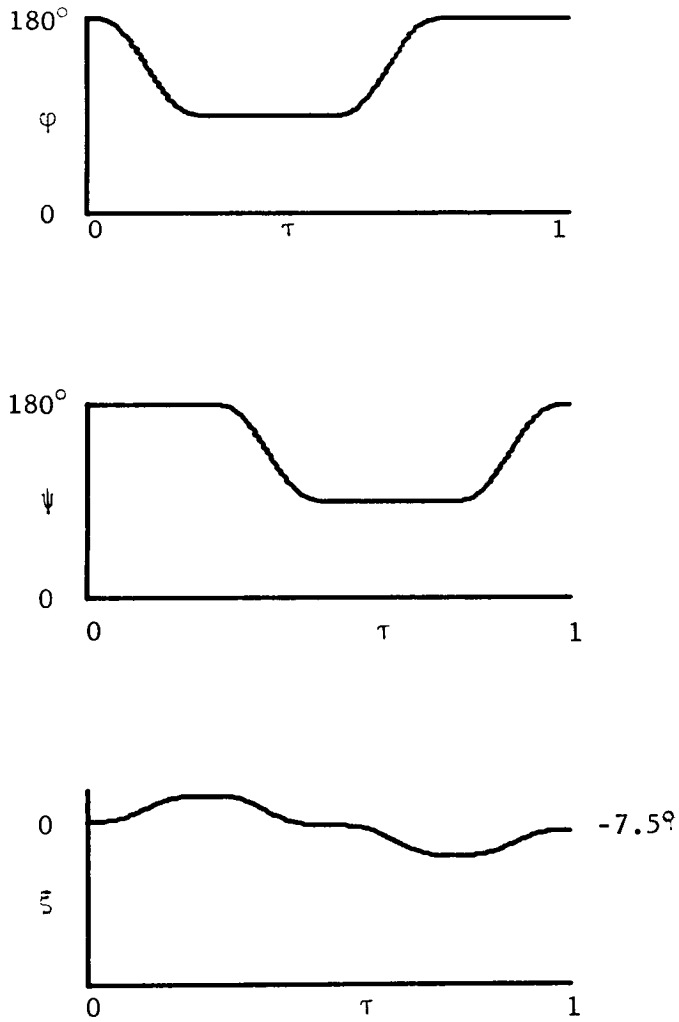

Fig. 4.5b

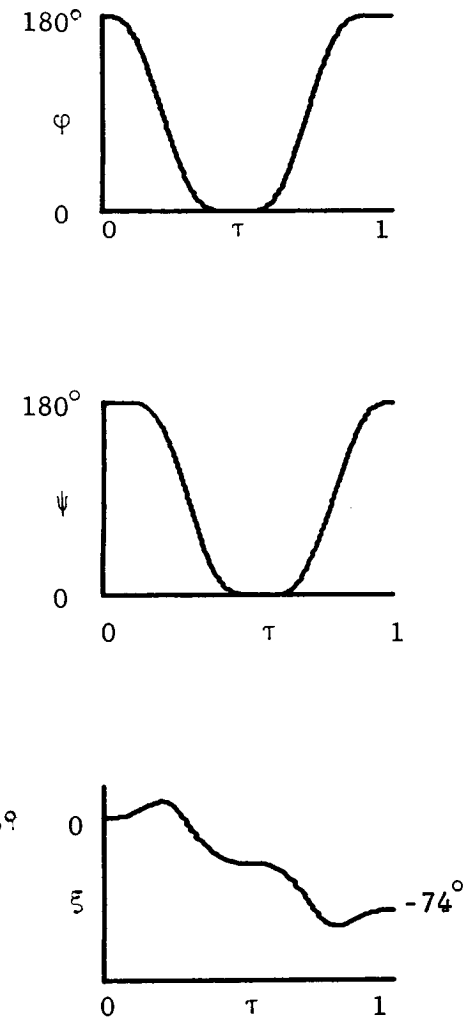

Fig. 4.5c

4.6 The constants in (4.2.7) can be chosen so that the model represents a human body. The reorientation maneuver can then be thought of as similar to that performed by a person executing a jackknife dive.

In the present context, body B_2 represents both arms acting as a unit. The legs are represented by B_1, and the remainder of the body, i.e., the trunk and the head, is represented by B_0. Appropriate inertia properties may be found by reference to data given for the fiftieth percentile of the United States Air Force flying population (or, more simply, the Air Force mean man) by Hanavan [25,26], which leads to

$$m = 161.9/32.2 \text{ slug} = 7.34 \times 10^4 \text{ gm}$$

$$m_1 = 53.0/32.2 \text{ slug} = 2.41 \times 10^4 \text{ gm}$$

$$m_2 = 18.4/32.2 \text{ slug} = 8.36 \times 10^3 \text{ gm}$$

$$a_0 = 10.5/12 \text{ ft} = 26.7 \text{ cm}$$

$$a_1 = 16.3/12 \text{ ft} = 41.4 \text{ cm}$$

$$a_2 = 10.4/12 \text{ ft} = 26.4 \text{ cm}$$

$$b_0 = 20.5/12 \text{ ft} = 52.1 \text{ cm}$$

$$I_0 = 1.56 \times 10^4/(32.2 \times 144) \text{ slug - ft}^2 = 4.57 \times 10^7 \text{ gm - cm}^2$$

$$I_1 = 4.65 \times 10^3/(32.2 \times 144) \text{ slug - ft}^2 = 1.36 \times 10^7 \text{ gm - cm}^2$$

$$I_2 = 1.13 \times 10^3/(32.2 \times 144) \text{ slug - ft}^2 = 3.31 \times 10^6 \text{ gm - cm}^2$$

The associated dimensionless parameters, (4.2.9), are

$$H_0 = .229 \quad , \quad H_1 = .0684 \quad , \quad H_2 = .0166$$

$$p_0 = .512 \quad , \quad p_1 = .795 \quad , \quad p_2 = .507$$

$$n_1 = .327 \quad , \quad n_2 = .114$$

Consider a maneuver in which the internal variables change one at a time according to the standard time history. In accordance with standard diving form, the maneuver begins and ends with the body in the "open"

position, i.e., $\varphi = \psi = 180^\circ$, and the movements of the legs precede those of the arms. Two parameters, φ^* and ψ^*, the intermediate values of φ and ψ, remain to be chosen. Fig. 4.6 gives the reorientation obtained for various values of φ^* and ψ^* which are physically

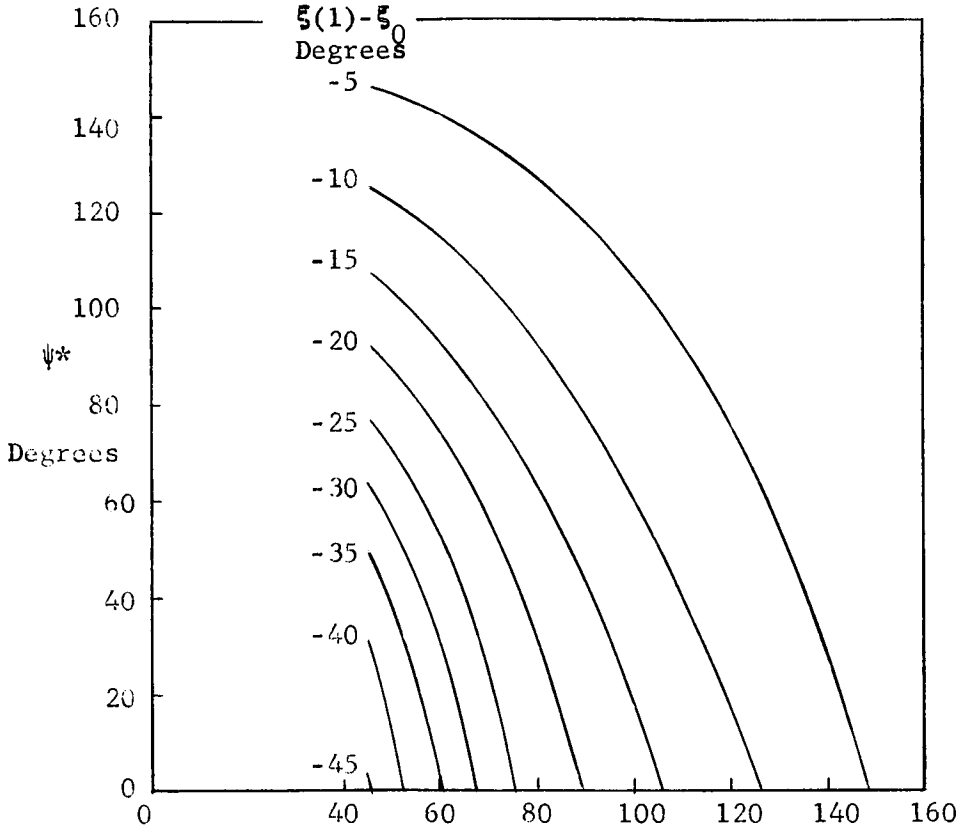

Fig. 4.6

attainable. Figure 14-2 of reference [42] shows that a champion diver attains a position corresponding to $\varphi^* = 50^\circ$, $\psi^* = 90^\circ$, and the associated reorientation, from Fig. 4.6, is about -19 degrees. The discrepancy between the diver's actual reorientation of +180 degrees and the predicted reorientation of -19 degrees is due to the diver not starting from rest as assumed in the analysis. One can see that a diver's angular momentum is indeed nonzero by comparing Figures 14-2 and 15-2 of reference [42].

5. Two Bodies in General Motion

5.1 A system, S, composed of two bodies, B_0 and B_1, of mass m_0 and m_1, is considered in this section. The bodies have moments of inertia I_0, J_0, K_0 and I_1, J_1, K_1 with respect to the principal axes X_0, Y_0, Z_0 and X_1, Y_1, Z_1 passing through the respective mass centers C_0 and C_1. As illustrated in Fig. 5.1, B_0 and B_1 are connected by a ball and socket joint at a point 0. This point is separated from the

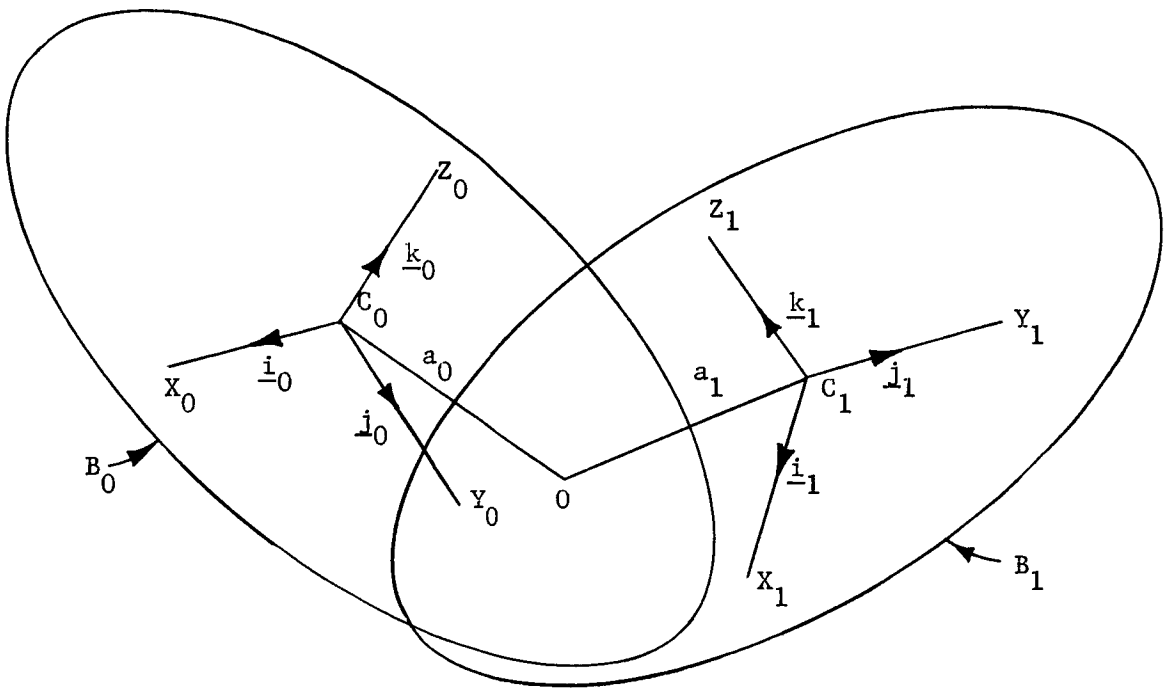

Fig. 5.1

two mass centers by distances a_0 and a_1, and the direction cosines of C_0O and C_1O are α_0, β_0, γ_0 and α_1, β_1, γ_1, respectively. The orientation of the principal axes of B_1 relative to those of B_0 depends on three internal variables, φ, ψ, and θ, these being two-axis Euler angles measured about the "Z", "X", and "Z" axes, respectively

(see Appendix 2). Instead of using the external variables, ξ, η, and ζ, in obtaining the equations of motion, the motion of B_0 is expressed in terms of the angular rates $\omega_0^{(1)}$, $\omega_o^{(2)}$, and $\omega_0^{(3)}$.

5.2 The equations of motion are obtained by computing the angular momentum of S with respect to the mass center, C*, of S and setting the scalar products of this quantity with three mutually perpendicular unit vectors equal to zero:

$$\underline{A}^{S/C^*} \cdot \underline{i}_0 = 0 \quad , \quad \underline{A}^{S/C^*} \cdot \underline{j}_0 = 0 \quad , \quad \underline{A}^{S/C^*} \cdot \underline{k}_0 = 0 \qquad (5.2.1)$$

Now,

$$\underline{A}^{S/C^*} = \underline{A}^{B_0/C_0} + \underline{A}^{B_1/C_1} + \underline{A}^{C_0/C^*} + \underline{A}^{C_1/C^*} \qquad (5.2.2)$$

where

$$\underline{A}^{B_0/C_0} = I_0 \, \omega_0^{(1)} \, \underline{i}_0 + J_0 \, \omega_0^{(2)} \, \underline{j}_0 + K_0 \, \omega_0^{(3)} \, \underline{k}_0 \qquad (5.2.3)$$

$$\underline{A}^{B_1/C_1} = I_1 \, \omega_1^{(1)} \, \underline{i}_1 + J_1 \, \omega_1^{(2)} \, \underline{j}_1 + K_1 \, \omega_1^{(3)} \, \underline{k}_1 \qquad (5.2.4)$$

$$\underline{A}^{C_0/C^*} = \underline{r}_0 \times m_0 \frac{d}{dt} \underline{r}_0 \qquad (5.2.5)$$

$$\underline{A}^{C_1/C^*} = \underline{r}_1 \times m_1 \frac{d}{dt} \underline{r}_1 \qquad (5.2.6)$$

The vectors $\underline{r}_0$ and $\underline{r}_1$ denote the position of C_0 and C_1, respectively, relative to C*:

$$\underline{r}_0 = \frac{m_1}{m_0 + m_1} \left[a_1(\alpha_1 \, \underline{i}_1 + \beta_1 \, \underline{j}_1 + \gamma_1 \, \underline{k}_1) - a_0(\alpha_0 \, \underline{i}_0 + \beta_0 \, \underline{j}_0 + \gamma_0 \, \underline{k}_0) \right] \qquad (5.2.7)$$

$$\underline{r}_1 = \frac{m_0}{m_0 + m_1} \left[a_0(\alpha_0 \, \underline{i}_0 + \beta_0 \, \underline{j}_0 + \gamma_0 \, \underline{k}_0) - a_1(\alpha_1 \, \underline{i}_1 + \beta_1 \, \underline{j}_1 + \gamma_1 \, \underline{k}_1) \right] \qquad (5.2.8)$$

Equations (5.2.2) - (5.2.8) together with the expressions for the angular rates of B_1 relative to B_0, (A-2.2), and the relations between the two sets of unit vectors, (A-2.1), given in Appendix 2 are sufficient for the determination of the equations of motion. These equations are†

$$[H_0^{(1)} + D_2\beta_0 + D_3\gamma_0 + E_{11}(c_\varphi c_\theta - c_\psi s_\varphi s_\theta) + E_{12}(- s_\theta c_\varphi - c_\psi c_\theta s_\varphi) + E_{13}s_\psi s_\varphi]\ \omega_0^{(1)} + [- D_2\alpha_0 + E_{11}(c_\theta s_\varphi + c_\psi s_\theta c_\varphi) + E_{12}(-s_\theta s_\varphi + c_\psi c_\theta c_\varphi) - E_{13}s_\psi c_\varphi]\ \omega_0^{(2)} + [- D_3\alpha_0 + E_{11}s_\psi s_\theta + E_{12}s_\psi c_\theta + E_{13}c_\psi]\ \omega_0^{(3)} = [-E_{11}s_\theta s_\psi - E_{12}c_\theta s_\psi - E_{13}c_\psi]\ \dot{\varphi} + [-E_{11}c_\theta + E_{12}s_\theta]\ \dot{\psi} - E_{13}\dot{\theta} \quad (5.2.9)$$

$$[-D_1\beta_0 + E_{21}(c_\varphi c_\theta - c_\psi s_\varphi s_\theta) + E_{22}(- s_\theta c_\varphi - c_\psi c_\theta s_\varphi) + E_{23}s_\psi s_\varphi]\ \omega_0^{(1)} + [H_0^{(2)} + D_3\gamma_0 + D_1\alpha_0 + E_{21}(c_\theta s_\varphi + c_\psi s_\theta c_\varphi) + E_{22}(- s_\theta s_\varphi + c_\psi c_\theta c_\varphi) - E_{23}s_\psi c_\varphi]\ \omega_0^{(2)} + [- D_3\beta_0 + E_{21}s_\psi s_\theta + E_{22}s_\psi c_\theta + E_{23}c_\psi]\ \omega_0^{(3)} = [-E_{21}s_\theta s_\psi - E_{22}c_\theta s_\psi - E_{23}c_\psi]\ \dot{\varphi} + [-E_{21}c_\theta + E_{22}s_\theta]\ \dot{\psi} - E_{23}\ \dot{\theta} \quad (5.2.10)$$

$$[-D_1\gamma_0 + E_{31}(c_\varphi c_\theta - c_\psi s_\varphi s_\theta) + E_{32}(- s_\theta c_\varphi - c_\psi c_\theta s_\varphi) + E_{33}s_\psi s_\varphi]\ \omega_0^{(1)} + [-D_2\gamma_0 + E_{31}(c_\theta s_\varphi + c_\psi s_\theta c_\varphi) + E_{32}(-s_\theta s_\varphi + c_\psi c_\theta c_\varphi) - E_{33}s_\psi c_\varphi]\ \omega_0^{(2)} + [H_0^{(3)} + D_1\alpha_0 + D_2\beta_0 + E_{31}s_\psi s_\theta + E_{32}s_\psi c_\theta + E_{33}c_\psi]\ \omega_0^{(3)} = [- E_{31}s_\theta s_\psi - E_{32}c_\theta s_\psi - E_{33}c_\psi]\ \dot{\varphi} + [- E_{31}c_\theta + E_{32}s_\theta]\ \dot{\psi} - E_{33}\ \dot{\theta} \quad (5.2.11)$$

† The lower case letters s and c denote sine and cosine functions, e.g., $c_\varphi = \cos\varphi$.

where

$$D_1 = \alpha_0 - p\alpha_1(c_\theta c_\varphi - c_\psi s_\theta s_\varphi) - p\beta_1(- s_\theta c_\varphi - c_\psi c_\theta s_\varphi) - p\gamma_1 s_\psi s_\varphi$$

$$D_2 = \beta_0 - p\alpha_1 (c_\theta s_\varphi + c_\psi s_\theta c_\varphi) - p\beta_1 (- s_\theta s_\varphi + c_\psi c_\theta c_\varphi) + p\gamma_1 s_\psi c_\varphi$$

$$D_3 = \gamma_0 - p\alpha_1 s_\psi s_\theta - p\beta_1 s_\psi c_\theta - p\gamma_1 c_\psi \qquad (5.2.12)$$

and

$$E_{11} = H_1^{(1)} (c_\theta c_\varphi - c_\psi s_\theta s_\varphi) + D_2 p(\gamma_1 s_\psi c_\theta - \beta_1 c_\psi) + D_3 p[\gamma_1 (s_\theta s_\varphi - c_\psi c_\theta c_\varphi) - \beta_1 s_\psi c_\varphi]$$

$$E_{12} = - H_1^{(2)} (s_\theta c_\varphi + c_\psi c_\theta s_\varphi) + D_2 p(-\gamma_1 s_\psi s_\theta + \alpha_1 c_\psi) + D_3 p[\gamma_1 (c_\theta s_\varphi + c_\psi s_\theta c_\varphi) + \alpha_1 s_\psi c_\varphi]$$

$$E_{13} = H_1^{(3)} s_\psi s_\varphi + D_2 p(\beta_1 s_\psi s_\theta - \alpha_1 s_\psi c_\theta) - D_3 p[\beta_1 (c_\theta s_\varphi + c_\psi s_\theta c_\varphi) + \alpha_1 (s_\theta s_\varphi - c_\psi c_\theta c_\varphi)]$$

$$E_{21} = H_1^{(1)} (c_\theta s_\varphi + c_\psi s_\theta c_\varphi) - D_3 p[\gamma_1 (s_\theta c_\varphi + c_\psi c_\theta s_\varphi) + \beta_1 s_\psi s_\varphi] + D_1 p(- \gamma_1 s_\psi c_\theta + \beta_1 c_\psi)$$

$$E_{22} = H_1^{(2)} (- s_\theta s_\varphi + c_\psi c_\theta c_\varphi) + D_3 p[\gamma_1 (- c_\theta c_\varphi + c_\psi s_\theta s_\varphi) + \alpha_1 s_\psi s_\varphi] + D_1 p(\gamma_1 s_\psi s_\theta - \alpha_1 c_\psi)$$

$$E_{23} = - H_1^{(3)} s_\psi c_\varphi + D_3 p[\beta_1 (c_\theta c_\varphi - c_\psi s_\theta s_\varphi) + \alpha_1 (s_\theta c_\varphi + c_\psi c_\theta s_\varphi)] + D_1 p(-\beta_1 s_\psi s_\theta + \alpha_1 s_\psi c_\theta)$$

$$E_{31} = H_1^{(1)}\, s_\psi s_\theta + D_1 p[\gamma_1(-\, s_\theta s_\varphi + c_\psi c_\theta c_\varphi) + \beta_1 s_\psi c_\varphi] + D_2 p[\gamma_1(s_\theta c_\varphi$$

$$+ c_\psi c_\theta s_\varphi) + \beta_1 s_\psi s_\varphi]$$

$$E_{32} = H_1^{(2)}\, s_\psi c_\theta - D_1 p[\gamma_1(c_\theta s_\varphi + c_\psi s_\theta c_\varphi) + \alpha_1 s_\psi c_\varphi] + D_2 p[\gamma_1(c_\theta c_\varphi$$

$$- c_\psi s_\theta s_\varphi) - \alpha_1 s_\psi s_\varphi]$$

$$E_{33} = H_1^{(3)}\, c_\psi + D_1 p[\beta_1(c_\theta s_\varphi + c_\psi s_\theta c_\varphi) + \alpha_1(s_\theta s_\varphi - c_\psi c_\theta c_\varphi)]$$

$$- D_2 p[\beta_1(c_\theta c_\varphi - c_\psi s_\theta s_\varphi) + \alpha_1(s_\theta c_\varphi + c_\psi c_\theta s_\varphi)] \qquad (5.2.13)$$

and where

$$p = \frac{a_1}{a_0}$$

$$H_i^{(1)} = \frac{I_i(m_0 + m_1)}{a_0^2 m_0 m_1}$$

$$H_i^{(2)} = \frac{J_i(m_0 + m_1)}{a_0^2 m_0 m_1}$$

$$H_i^{(3)} = \frac{K_i(m_0 + m_1)}{a_0^2 m_0 m_1} \quad , \quad i = 0,1 \qquad (5.2.14)$$

When using (5.2.9) - (5.2.11), the angular rates $\omega_0^{(1)}$, $\omega_0^{(2)}$, and $\omega_0^{(3)}$ may be expressed in terms of the external variables ξ, η, and ζ by using the appropriate kinematical equations in Appendix 2.

5.3 The equations above apply to a general two-body system. For two uniform thin rods connected at their ends, the constants in the equations have the values

$$\alpha_0 = \alpha_1 = -1 \quad , \quad \beta_0 = \beta_1 = \gamma_0 = \gamma_1 = 0$$

$$H_0^{(1)} = H_1^{(1)} = 0 \quad , \quad H_0^{(2)} = H_0^{(3)} = \frac{1}{3p}(1+p) \quad , \quad H_1^{(2)} = H_1^{(3)} = \frac{p^2}{3}(1+p) \tag{5.3.1}$$

which depend only on p, the length ratio of the rods. In addition, suppose that the restriction $\psi = -90^\circ$ is placed on the relative motion. Equations (5.2.9) - (5.2.13) now reduce to

$$-(E_{12}s_\theta c_\varphi + E_{13}s_\varphi)\,\omega_0^{(1)} - (-D_2 + E_{12}s_\theta s_\varphi - E_{13}c_\varphi)\,\omega_0^{(2)} - (-D_3 + E_{12}c_\theta)\,\omega_0^{(3)} - E_{12}c_\theta\,\dot{\varphi} + E_{13}\,\dot{\theta} = 0 \tag{5.3.2}$$

$$-(E_{22}s_\theta c_\varphi + E_{23}s_\varphi)\,\omega_0^{(1)} + (H_0^{(2)} - D_1 - E_{22}s_\theta s_\varphi + E_{23}c_\varphi)\,\omega_0^{(2)} - E_{22}c_\theta\omega_0^{(3)} - E_{22}c_\theta\,\dot{\varphi} + E_{23}\,\dot{\theta} = 0 \tag{5.3.3}$$

$$-(E_{32}s_\theta c_\varphi + E_{33}s_\varphi)\,\omega_0^{(1)} - (E_{32}s_\theta s_\varphi - E_{33}c_\varphi)\,\omega_0^{(2)} + (H_0^{(2)} - D_1 - E_{32}c_\theta)\,\omega_0^{(3)} - E_{32}c_\theta\,\dot{\varphi} + E_{33}\,\dot{\theta} = 0 \tag{5.3.4}$$

where

$$D_1 = -1 + p\,c_\theta c_\varphi$$

$$D_2 = p\,c_\theta s_\varphi$$

$$D_3 = -p\,s_\theta \tag{5.3.5}$$

and

$$E_{12} = -H_1^{(2)}\,s_\theta c_\varphi + p\,D_3\,c_\varphi$$

$$E_{13} = -H_1^{(2)}\,s_\varphi - p\,D_2 c_\theta + p\,D_3 s_\theta s_\varphi$$

$$E_{22} = - H_1^{(2)} s_\theta s_\varphi + p D_3 s_\varphi$$

$$E_{23} = H_1^{(2)} c_\varphi - p D_3 s_\theta c_\varphi + p D_1 c_\theta$$

$$E_{32} = - H_1^{(2)} c_\theta - p D_1 c_\varphi - p D_2 s_\varphi$$

$$E_{33} = - p D_1 s_\theta s_\varphi + p D_2 s_\theta c_\varphi \qquad (5.3.6)$$

The possible relative motions of the rods under the restriction $\psi = -90°$ may be understood more easily by imagining a set of mutually perpendicular axes oriented such that U_2 is directed along $\underline{k}_1$, U_3 along $\underline{k}_0$, and U_1 along $\underline{k}_1 \times \underline{k}_0$. As shown in Fig. 5.3, B_0 rotates

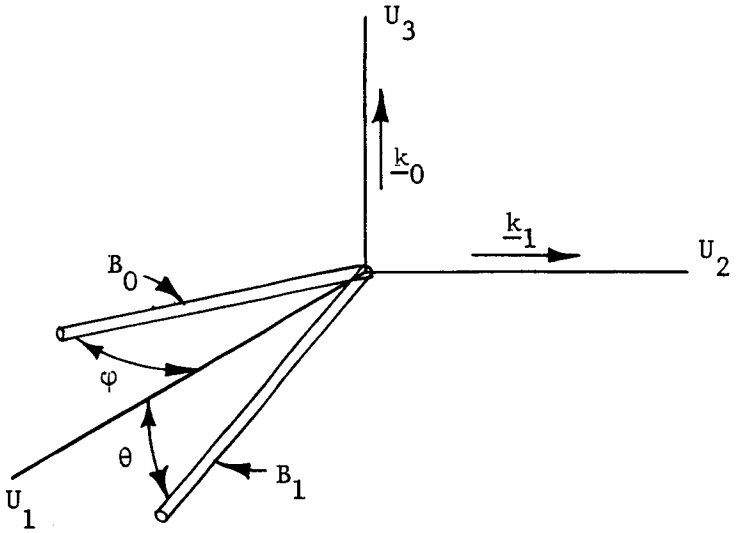

Fig. 5.3

about U_3 and B_1 rotates about U_2, and the angles between the rods and U_1 are φ and θ, respectively.

Let the rods be of equal length, and consider a four-part maneuver in which φ and θ change successively, as follows: φ increases from

zero to a value φ^*; next, θ increases from zero to θ^*; and φ and θ then individually return to zero.

Numerical integration of (5.3.2) - (5.3.4), using the standard time history for φ and θ, leads to results which suggest that the reorientation of the system is independent of φ^* and θ^*. Specifically, it is found that, when φ^* and θ^* both have the values 30, 60, 90, 120, and 150 degrees, the reorientation of the system is equivalent to a 90 degree rotation about the initial U_1 axis. For $\varphi^* = \theta^* = 90^\circ$, validity of this conclusion may be established by means of the following argument: when φ opens to 90 degrees, B_0 becomes parallel to U_2; consequently, as θ opens, the rotation about U_2 results only in a rotation of B_0 about its axis, since B_0 has no moment of inertia about its axis; as φ closes, B_1 likewise rotates 90 degrees about its axis; these rotations of the bodies about their axes account for the 90 degree reorientation. (One numerical integration of (5.2.9) - (5.2.11) for identical uniform thin rods and $\psi \equiv -30^\circ$ was attempted, and it yielded a reorientation equivalent to a 30 degree rotation about the initial X_0 axis, suggesting that the amount of reorientation is equal to the value of ψ.)

When φ^* and θ^*, are small, the 90 degree reorientation may be demonstrated by linearization of (5.3.2) - (5.3.4) in φ, θ, $\dot{\varphi}$, and $\dot{\theta}$; for $p = 1$ this yields

$$- \varphi \, \omega_0^{(2)} + \theta \, \omega_0^{(3)} = 0 \tag{5.3.7}$$

$$- \varphi \, \omega_0^{(1)} + 2 \, \omega_0^{(2)} = - \dot{\theta} \tag{5.3.8}$$

$$\theta \, \omega_0^{(1)} + 2 \, \omega_0^{(3)} = - \dot{\varphi} \tag{5.3.9}$$

The latter two of these show that $\omega_0^{(2)}$ and $\omega_0^{(3)}$ are small whenever φ, θ, $\dot{\varphi}$, and $\dot{\theta}$ are small. Solution of (5.3.7) - (5.3.9) leads to

$$\omega_0^{(1)} = \frac{\varphi\dot{\theta} - \theta\dot{\varphi}}{\varphi^2 + \theta^2} \tag{5.3.10}$$

which may be integrated over a four-part maneuver to give the total reorientation of the system. The reorientation, ξ_1, associated with the value of $\omega_0^{(1)}$ resulting from the first part of the maneuver is zero, since θ and $\dot{\theta}$ are zero. Similarly, the reorientation during the fourth part, ξ_4, is zero. Supposing that θ increases linearly, i.e., $\theta = \frac{\theta^*}{T}\, t$, where T is the time required for the second part of the maneuver, one may compute ξ_2 by integrating (5.3.10):

$$\xi_2 = \int_0^T \frac{\varphi^*\theta^*/T}{\varphi^{*2} + (\frac{\theta^*}{T}\, t)^2}\, dt = \tan^{-1}\frac{\theta^*}{\varphi^*} \tag{5.3.11}$$

Likewise, if φ decreases linearly from φ^*, and T represents the duration of the third part of the maneuver, then

$$\xi_3 = \int_0^T \frac{\varphi^*\theta^*/T}{\varphi^{*2}(1 - \frac{t}{T})^2 + \theta^{*2}}\, dt = \frac{\theta^* T}{\varphi^*} \int_{-T}^{0} \frac{d\,\sigma}{\sigma^2 + (\theta^* T/\varphi^*)^2} = \tan^{-1}\frac{\varphi^*}{\theta^*} \tag{5.3.12}$$

where $\sigma = t - T$. The total reorientation is thus

$$\xi_2 + \xi_3 = \tan^{-1}\frac{\theta^*}{\varphi^*} + \tan^{-1}\frac{\varphi^*}{\theta^*} = 90^\circ \tag{5.3.13}$$

Hence, the 90 degree reorientation holds for any small, nonzero values of φ^* and θ^*.

The system under consideration has the interesting property of responding with large reorientations to small changes in the internal variables. In the next section, an attempt is made to modify the system so as to obtain a model satisfactorily representing the human body while

retaining this property. This involves making three changes: a third body, representing the human trunk, is introduced; a distance is added between the hinge points at the ends of the rods, the rods playing the parts of arms or legs; and allowance is made for the moments of inertia of the rods about their axes of symmetry.

6. Three Bodies in General Motion

6.1 Consider a system, S, composed of a main body, B_0, to which are attached by means of hinges two identical bodies B_1 and B_2. Unit vectors $\underline{i}_i$, $\underline{j}_i$, $\underline{k}_i$, $i = 0, 1, 2$, are aligned with the principal axes of the bodies, and, in addition, unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ are fixed in B_0 such that $\underline{i}$ and $\underline{j}$ are parallel to the hinge axes passing through O_1 and O_2, respectively, as shown in Fig. 6.1. Bodies B_1 and B_2 each

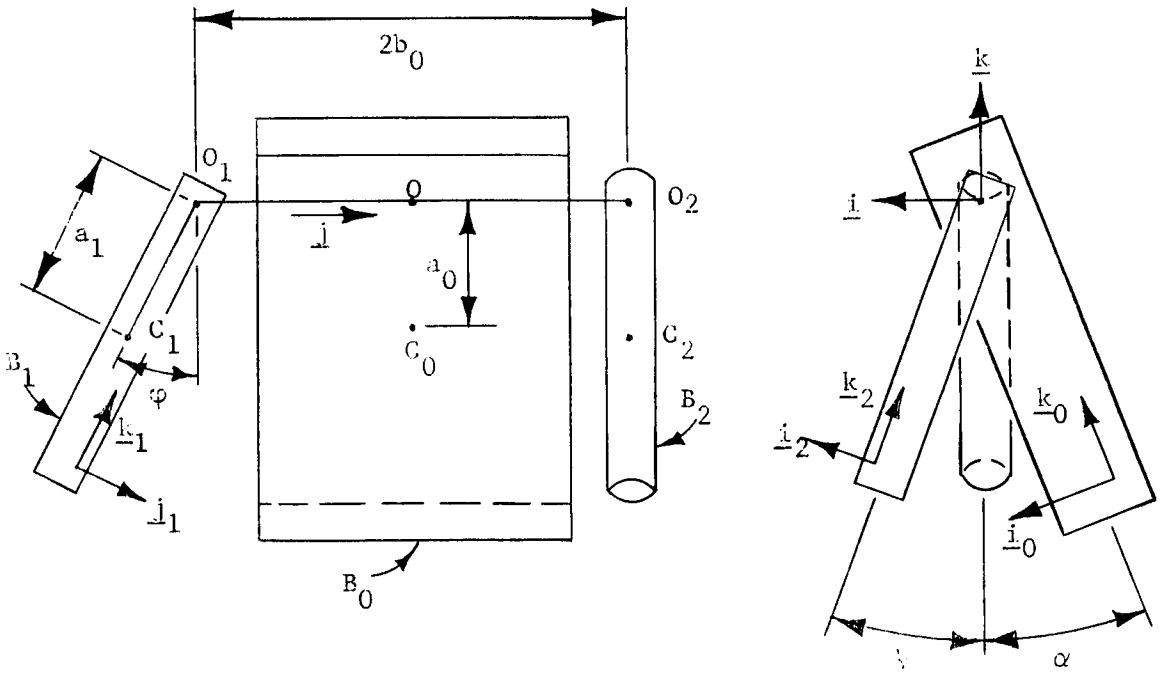

Fig. 6.1

have mass m_1 and principal moments of inertia I_1, I_1, and K_1, for their mass centers C_1 and C_2, and B_0 has mass m_0 and principal moments of inertia I_0, J_0, and K_0 for the mass center C_0. The internal variables are φ and ψ; however, instead of employing the external variables ξ, η, and ζ directly, the motion of B_0 is expressed in terms of the angular rates $\omega^{(1)}$, $\omega^{(2)}$, and $\omega^{(3)}$ referred to $\underline{i}$, $\underline{j}$, and $\underline{k}$.

6.2 Equations of motion are obtained by equating to zero the angular momentum,

$$\underline{A}^{S/C^*} = \underline{A}^{B_0/C_0} + \underline{A}^{B_1/C_1} + \underline{A}^{B_2/C_2} + \underline{A}^{C_0/C^*} + \underline{A}^{C_1/C^*} + \underline{A}^{C_2/C^*} \qquad (6.2.1)$$

of S with respect to C^*, the mass center of S. The first three terms of (6.2.1) are

$$\begin{aligned}
\underline{A}^{B_0/C_0} &= I_0\,\omega_0^{(1)}\,\underline{i}_0 + J_0\,\omega_0^{(2)}\,\underline{j}_0 + K_0\,\omega_0^{(3)}\,\underline{k}_0 \\
\underline{A}^{B_1/C_1} &= I_1\,\omega_1^{(1)}\,\underline{i}_1 + I_1\,\omega_1^{(2)}\,\underline{j}_1 + K_1\,\omega_1^{(3)}\,\underline{k}_1 \\
\underline{A}^{B_2/C_2} &= I_1\,\omega_2^{(1)}\,\underline{i}_2 + I_1\,\omega_2^{(2)}\,\underline{j}_2 + K_1\,\omega_2^{(3)}\,\underline{k}_2
\end{aligned} \qquad (6.2.2)$$

and the last three are

$$\begin{aligned}
\underline{A}^{C_0/C^*} &= m_0\,(\underline{r} - a_0\,\underline{k}_0) \times \frac{d}{dt}\,(\underline{r} - a_0\,\underline{k}_0) \\
\underline{A}^{C_1/C^*} &= m_1\,(\underline{r} - b_0\,\underline{j}_0 - a_1\,\underline{k}_1) \times \frac{d}{dt}\,(\underline{r} - b_0\,\underline{j}_0 - a_1\,\underline{k}_1) \\
\underline{A}^{C_2/C^*} &= m_1(\underline{r} + b_0\underline{j}_0 - a_1\underline{k}_2) \times \frac{d}{dt}\,(\underline{r} + b_0\underline{j}_0 - a_1\underline{k}_2)
\end{aligned} \qquad (6.2.3)$$

where $\underline{r}$ is the position vector of 0 relative to C^*:

$$\underline{r} = \frac{m_0 a_0}{m_0 + 2m_1}\,\underline{k}_0 + \frac{m_1 a_1}{m_0 + 2m_1}\,(\underline{k}_1 + \underline{k}_2) \qquad (6.2.4)$$

The following relationships are useful for expressing all quantities in (6.2.2) and (6.2.3) in terms of a common set of unit vectors and angular rates:

$$\underline{i}_0 = c_\alpha \underline{i} - s_\alpha \underline{k} \ , \quad \underline{j}_0 = \underline{j} \ , \quad \underline{k}_0 = s_\alpha \underline{i} + c_\alpha \underline{k}$$

$$\underline{i}_1 = \underline{i} \ , \quad \underline{j}_1 = c_\varphi \underline{j} - s_\varphi \underline{k} \ , \quad \underline{k}_1 = s_\varphi \underline{j} + c_\varphi \underline{k}$$

$$\underline{i}_2 = c_\psi \underline{i} + s_\psi \underline{k} \ , \quad \underline{j}_2 = \underline{j} \ , \quad \underline{k}_2 = -s_\psi \underline{i} + c_\psi \underline{k} \qquad (6.2.5)$$

$$\omega_0^{(1)} = c_\alpha \omega^{(1)} - s_\alpha \omega^{(3)} \ , \quad \omega_0^{(2)} = \omega^{(2)} \ , \quad \omega_0^{(3)} = s_\alpha \omega^{(1)} + c_\alpha \omega^{(3)}$$

$$\omega_1^{(1)} = \omega^{(1)} - \dot{\varphi} \ , \quad \omega_1^{(2)} = c_\varphi \omega^{(2)} - s_\varphi \omega^{(3)} \ , \quad \omega_1^{(3)} = s_\varphi \omega^{(2)} + c_\varphi \omega^{(3)}$$

$$\omega_2^{(1)} = c_\psi \omega^{(1)} + s_\psi \omega^{(3)} \ , \quad \omega_2^{(2)} = \omega^{(2)} - \dot{\psi} \ , \quad \omega_2^{(3)} = -s_\psi \omega^{(1)} + c_\psi \omega^{(3)} \qquad (6.2.6)$$

Eight dimensionless parameters are required to define the system:

$$H_0^{(1)} = \frac{I_0}{m a_0 a_1} \ , \quad H_0^{(2)} = \frac{J_0}{m a_0 a_1} \ , \quad H_0^{(3)} = \frac{K_0}{m a_0 a_1}$$

$$H_1^{(1)} = \frac{I_1}{m a_0 a_1} \ , \quad H_1^{(3)} = \frac{K_1}{m a_0 a_1}$$

$$p_1 = \frac{a_1}{a_0} \ , \quad p_2 = \frac{b_0}{a_0} \ , \quad n = \frac{m_0}{m_1} \qquad (6.2.7)$$

where m is the mass of S:

$$m = m_0 + 2m_1 \qquad (6.2.8)$$

It is convenient to introduce three additional quantities:

$$n_0 = \frac{n}{n+2} \ , \quad n_1 = \frac{1}{n+2} \ , \quad n_2 = \frac{n^2 + 2n + 2}{n^2 + 4n + 4} \qquad (6.2.9)$$

The equations of motion, obtained by carrying out the operations indicated in (6.2.1) - (6.2.3), making use of (6.2.4) - (6.2.9), and

simplifying, may be written as follows:

$$A_1\omega^{(1)} + A_2\omega^{(2)} + A_3\omega^{(3)} = (-E_1 + D_2)\,\dot{\varphi} - E_{19}s_\psi\,\dot{\psi} \qquad (6.2.10)$$

$$A_4\omega^{(1)} + A_5\omega^{(2)} + A_6\omega^{(3)} = E_{13}s_\varphi\,\dot{\varphi} - (E_1 - D_3)\,\dot{\psi} \qquad (6.2.11)$$

$$A_7\omega^{(1)} + A_8\omega^{(2)} + A_9\omega^{(3)} = E_{13}c_\varphi\,\dot{\varphi} + E_{19}c_\psi\,\dot{\psi} \qquad (6.2.12)$$

where

$$\begin{aligned}
A_1 &= D_1 + E_2 - E_1 + E_6c_\alpha + E_8s_\alpha + E_{16}c_\psi - E_{18}s_\psi \\
A_2 &= E_9s_\alpha - E_{19}s_\psi \\
A_3 &= E_7c_\alpha + E_{10}s_\alpha \\
A_4 &= E_3 + E_{13}s_\varphi \\
A_5 &= D_1 + E_4 + E_{11}c_\varphi + E_{14}s_\varphi \\
A_6 &= E_5 + E_{12}c_\varphi + E_{15}s_\varphi \\
A_7 &= -E_6s_\alpha + E_8c_\alpha + E_{13}c_\varphi + E_{16}s_\psi + E_{18}c_\psi \\
A_8 &= E_9c_\alpha - E_{11}s_\varphi + E_{14}c_\varphi + E_{19}c_\psi \\
A_9 &= D_1 + E_2 - E_7s_\alpha + E_{10}c_\alpha - E_{12}s_\varphi + E_{15}c_\varphi + E_{17}s_\psi + E_{20}c_\psi
\end{aligned} \qquad (6.2.13)$$

$$\begin{aligned}
D_1 &= 2n_1\left\{n_0n_1p_1 - n_0\left[c_\varphi c_\alpha + \cos(\psi+\alpha)\right] + p_2s_\varphi - n_1p_1c_\varphi c_\psi\right\} \\
D_2 &= -n_0n_1c_\varphi c_\alpha - n_1^2p_1c_\varphi c_\psi + n_0n_1^2p_1 + n_1p_2s_\varphi \\
D_3 &= -n_0n_1\cos(\psi+\alpha) - n_1^2p_1c_\varphi c_\psi + n_0n_1^2p_1
\end{aligned} \qquad (6.2.14)$$

$$\begin{aligned}
E_1 &= -H_1^{(1)} - n_1n_2p_1 \\
E_2 &= 2n_1p_2^2/p_1
\end{aligned}$$

$$E_3 = - n_1 p_2 s_\psi$$

$$E_4 = H_0^{(2)} + 2n_0 n_1/p_1 - n_1 p_2 s_\varphi + H_1^{(1)} + n_1 n_2 p_1$$

$$E_5 = n_1 p_2 \ (c_\psi - c_\varphi)$$

$$E_6 = (H_0^{(1)} + 2n_0 n_1/p_1) \ c_\alpha$$

$$E_7 = - (H_0^{(1)} + 2n_0 n_1/p_1) \ s_\alpha$$

$$E_8 = - n_0 n_1 s_\psi + H_0^{(3)} s_\alpha$$

$$E_9 = n_0 n_1 s_\varphi$$

$$E_{10} = n_0 n_1 \ (c_\varphi + c_\psi) + H_0^{(3)} c_\alpha$$

$$E_{11} = (H_1^{(1)} + n_1 n_2 p_1) \ c_\varphi$$

$$E_{12} = - (H_1^{(1)} + n_1 n_2 p_1) \ s_\varphi$$

$$E_{13} = - n_1^2 p_1 s_\psi + n_0 n_1 s_\alpha$$

$$E_{14} = - n_1 p_2 + (H_1^{(3)} - n_0 n_1^2 p_1) \ s_\varphi$$

$$E_{15} = n_1^2 p_1 c_\psi + n_0 n_1 c_\alpha + (H_1^{(3)} - n_0 n_1^2 p_1) \ c_\varphi$$

$$E_{16} = (H_1^{(1)} + n_1 n_2 p_1) \ c_\psi$$

$$E_{17} = (H_1^{(1)} + n_1 n_2 p_1) \ s_\psi$$

$$E_{18} = n_0 n_1 s_\alpha - (H_1^{(3)} - n_0 n_1^2 p_1) \ s_\psi$$

$$E_{19} = n_1^2 p_1 s_\varphi + n_1 p_2$$

$$E_{20} = n_0 n_1 c_\alpha + n_1^2 p_1 c_\varphi + (H_1^{(3)} - n_0 n_1^2 p_1) \ c_\psi \qquad (6.2.15)$$

6.3 The present system may be used as a model of the human body in two ways: either B_1 and B_2 are the arms, and B_0 is the remainder of the body; or B_1 and B_2 are the legs, and the remainder is B_0. the reorientation of the body due to motion of the arms is examined in this subsection by solving (6.2.10) - (6.2.12) numerically for several prescribed motions of the arms; the same procedure is then followed in the next subsection for the legs. The Algol computer program used for this work is listed in Appendix 3.

The required anthropometric data are obtained from information given by Hanavan [25,26] for the Air Force mean man:

$m = 161.9/32.2$ slug $= 7.34 \times 10^4$ gm

$m_1 = 9.2/32.2$ slug $= 4.18 \times 10^3$ gm

$a_0 = 17.8/12$ ft $= 45.2$ cm

$a_1 = 10.4/12$ ft $= 26.4$ cm

$b_0 = 8.0/12$ ft $= 20.3$ cm

$I_0 = 3.86 \times 10^4/(32.2 \times 144)$ slug - ft^2 $= 1.13 \times 10^8$ gm - cm^2

$J_0 = 3.77 \times 10^4/(32.2 \times 144)$ slug - ft^2 $= 1.10 \times 10^8$ gm - cm^2

$K_0 = 1.92 \times 10^3/(32.2 \times 144)$ slug - ft^2 $= 5.62 \times 10^6$ gm - cm^2

$I_1 = 565/(32.2 \times 144)$ slug - ft^2 $= 1.66 \times 10^6$ gm - cm^2

$K_1 = 14/(32.2 \times 144)$ slug - ft^2 $= 4.10 \times 10^4$ gm - cm^2

The dimensionless parameters defined in (6.2.7) have the values

$$H_0^{(1)} = 1.287 \quad , \quad H_0^{(2)} = 1.256 \quad , \quad H_0^{(3)} = .0641$$

$$H_1^{(1)} = .0188 \quad , \quad H_1^{(3)} = .00047$$

$$p_1 = .584 \quad , \quad p_2 = .446 \quad , \quad n = 15.6$$

In the maneuver to be considered, both arms are at the sides of the body, i.e., $\varphi = \psi = \alpha = 0$, at the beginning and at the end of the motion; and, as before, the maneuver takes place in four parts. The right arm (B_1) swings to the side to form an angle φ^* with the body; next, the left arm swings forward to an angle ψ^*; finally, both arms return to the sides of the body, the right arm moving first. The reorientation produced by such a maneuver, expressed in terms of the three-axis Euler angles (Appendix 2) ξ, η, and ζ involving "eye," "shoulder," and "spine" axes,† respectively, depends on the values of φ^* and ψ^* as shown in Fig. 6.3.

For given values of φ^* and ψ^*, the reorientation may be found from this plot by first locating the point which corresponds to these values and then reading η on the abscissa, ζ on the ordinate, and ξ (which is generally smallest) on the field. As an example, consider a maneuver in which the right arm swings to the side and the left one swings forward, the two arms forming respective angles with the spine axis of 90 degrees and 30 degrees, i.e., $\varphi^* = 90^\circ$ and $\psi^* = 30^\circ$. The corresponding external variables, $\xi = .002^\circ$ (approximately), $\eta = -.3^\circ$, and $\zeta = -2.6^\circ$, may be read from Fig. 6.3. (Note that the maneuver associated with $\varphi^* = 150^\circ$ and $\psi^* = 180^\circ$ yield similar values of η and ζ, but a much larger value of ξ than in the original case, namely $\xi = .8^\circ$.) This reorientation may be thought of as a 2-1/2 degree right-handed rotation about the "down" spine axis which itself rotates so that the body "leans back" one third of a degree. As discussed in Appendix 2,

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† These terms are meant to suggest axes parallel to $\underline{i}_0$, $\underline{j}$, and $\underline{k}_0$, respectively.

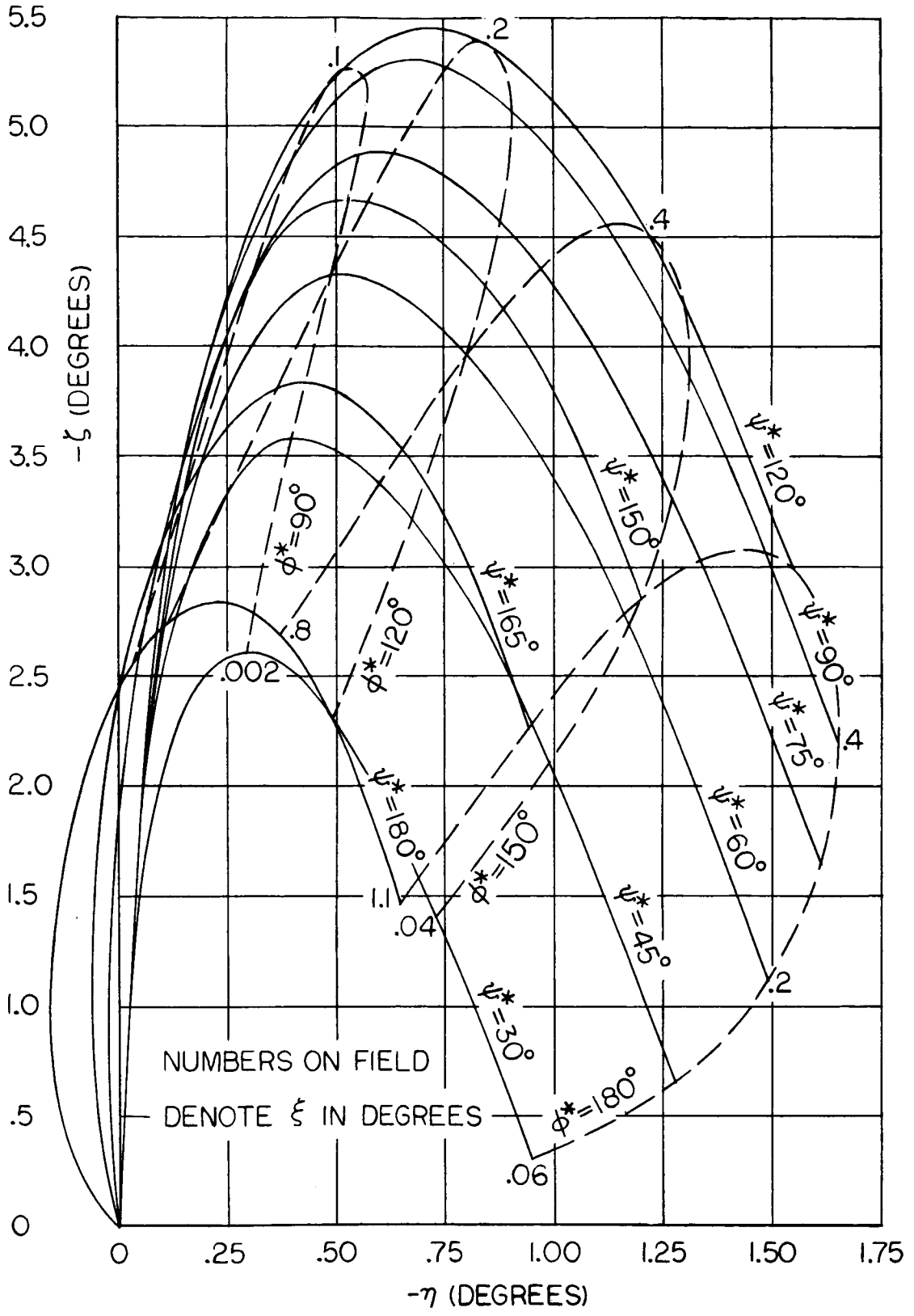
5.5
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
.5
0
-ζ (DEGREES)
0
.25
.50
.75
1.00
1.25
1.50
1.75
-η (DEGREES)
.1
.2
.4
.8
.002
1.1
.04
.06
.4
.2
ψ*=120°
ψ*=150°
ψ*=90°
ψ*=165°
ψ*=75°
ψ*=180°
ψ*=60°
ψ*=45°
ψ*=30°
φ*=90°
φ*=120°
φ*=150°
φ*=180°
NUMBERS ON FIELD
DENOTE ξ IN DEGREES

Fig. 6.3

this interpretation of the reorientation, expressed as three-axis Euler angles, is possible because the Euler angles themselves are small.

The angle α determines the initial position of the arms in relation to the trunk. For example, $\alpha = 90^{\circ}$ places the arms in front of the body, parallel to the eye axis, at the beginning of the maneuver. Since the model considered in Section 5.3 produces a reorientation equivalent to a rotation about the initial direction of the rods, and since Fig. 6.3 shows that the reorientation is predominantly a rotation about the spine axis when $\alpha = 0$, one might expect that the reorientation would be predominantly a rotation about the eye axis if $\alpha = 90^{\circ}$. However, numerical integrations for this case show that the reorientation is still essentially a rotation about the spine axis, apparently because the moments of inertia associated with this axis are much smaller than those associated with the eye axis.

6.4 One can achieve larger reorientations about the spine axis by moving the legs than by moving the arms. The three-body model may be used to demonstrate this by taking $\alpha = 180^{\circ}$ and setting

$m = 161.9/32.2$ slug $= 7.34 \times 10^4$ gm

$m_1 = 26.5/32.2$ slug $= 1.20 \times 10^4$ gm

$a_0 = 10.2/12$ ft $= 25.9$ cm

$a_1 = 16.3/12$ ft $= 41.4$ cm

$b_0 = 3.0/12$ ft $= 7.6$ cm

$I_0 = 1.16 \times 10^4/(32.2 \times 144)$ slug - ft^2 $= 3.40 \times 10^7$ gm - cm^2

$J_0 = 8.00 \times 10^3/(32.2 \times 144)$ slug - ft^2 $= 2.34 \times 10^7$ gm - cm^2

$K_0 = 2.43 \times 10^3/(32.2 \times 144)$ slug - ft^2 $= 7.12 \times 10^6$ gm - cm^2

$I_1 = 2.32 \times 10^3/(32.2 \times 144)$ slug - ft^2 $= 6.79 \times 10^6$ gm - cm^2

$K_1 = 100/(32.2 \times 144)$ slug - ft^2 $= 2.93 \times 10^5$ gm - cm^2

The eight dimensionless parameters (6.2.7) are

$$H_0^{(1)} = .431 \quad , \quad H_0^{(2)} = .297 \quad , \quad H_0^{(3)} = .0903$$

$$H_1^{(1)} = .0864 \quad , \quad H_1^{(3)} = .00372$$

$$p_1 = 1.60 \quad , \quad p_2 = .294 \quad , \quad n = 4.12$$

The maneuver commences with both legs down ($\varphi = \psi = 0$), and takes place in four parts: the right leg swings outward to φ^*; the left leg swings forward to ψ^*; the right leg returns to its initial position; and the left leg returns to the initial position. The reorientation of the system is measured by means of the three-axis Euler angles ξ, η, and ζ, whose dependence on φ^* and ψ^* is illustrated in Fig. 6.4. By comparing this plot with Fig. 6.3, one can see that use of the legs results in reorientations which are not only larger, but which also are more nearly equivalent to pure rotations about a spine axis, even though φ^* and ψ^* are here restricted to values smaller than those used for the arm motions.

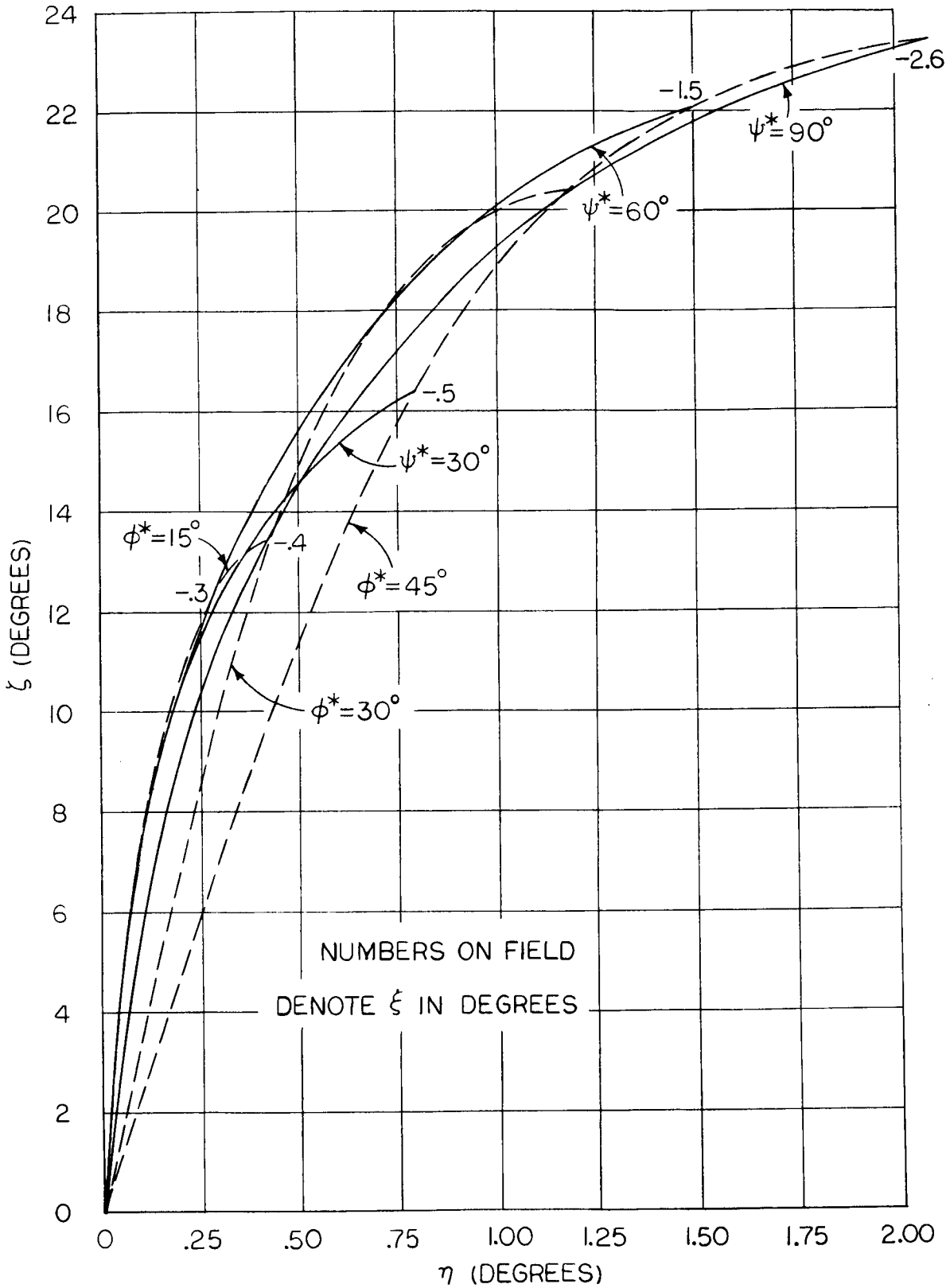
24
22
20
18
16
14
12
10
8
6
4
2
0
ζ (DEGREES)
0
.25
.50
.75
1.00
1.25
1.50
1.75
2.00
η (DEGREES)
-2.6
-1.5
ψ*=90°
ψ*=60°
-.5
ψ*=30°
φ*=15°
-.4
-.3
φ*=45°
φ*=30°
NUMBERS ON FIELD
DENOTE ξ IN DEGREES

Fig. 6.4

III. THE SECOND PROBLEM

7. General Formulation and Method of Solution

7.1 The Second Problem, as explained in Section 2.2, is that of finding a set of functions which, when used as internal variables, describe a maneuver in which the system moves from a specified initial state to a specified final state. A general mathematical statement of the Second Problem, as well as an outline of the method of solution, is given in this section, and an example of the application of this material to a specific problem is given in Section 8.

Equations of motion for a system composed of connected rigid bodies which fall freely after starting from rest may be written

$$P(\Phi)\ \Phi' + Q(\Xi)\ \Xi' = 0 \qquad (7.1.1)$$

where P and Q are, respectively, $r \times s$ and $r \times r$ matrices of coefficients; Φ and Ξ are column vectors of internal and external variables, respectively; r is the number of external variables ($r = 1$ for planar motion and $r = 3$ for general motion), and s is the number of internal variables. To specify the initial ($\tau = 0$) and final ($\tau = 1$) states of the system, $2(r + s)$ boundary conditions are required:†

$$\Phi(0) = \Phi_0 \quad , \quad \Phi(1) = \Phi_n$$
$$\Xi(0) = \Xi_0 \quad , \quad \Xi(1) = \Xi_n \qquad (7.1.2)$$

The boundary value problem (7.1.1), (7.1.2) is underdetermined since the

† The reason for the choice of n as a subscript will become apparent in Section 8.2.

number of unknowns in (7.1.1) exceeds the number of equations. Consequently, (7.1.1) and (7.1.2) cannot be expected to possess unique solutions, or, expressed in terms of motions, there may exist more than one sequence of relative motions of system components for which the system moves from a specified initial state to a specified final state.

7.2 The underdetermined problem is solved by adding equations to (7.1.1) in a number sufficient to render the problem solvable by techniques normally used on boundary value problems. The resulting system of equations must not only be a determined system, but must also be of order $2(r + s)$ so that (7.1.2) may be satisfied.

Additional equations are obtained by imposing the requirement that the solution be optimal with respect to some criterion. Specifically, of all the solutions of (7.1.1) which satisfy (7.1.2), that one is sought which renders J stationary, where

$$J\ [\Phi,\Pi] = \int_0^1 G\ (\Phi,\Phi',\Pi,\Pi')\ d\tau \qquad (7.2.1)$$

and where G is any desired function of its arguments.

The problem just stated is known in the calculus of variations as a Lagrange problem or as a problem with nonholonomic constraints. (In this context, (7.1.1) is called the constraint equation.) It is treated extensively by Bliss [43], and there exist some more recent accounts, e.g., Akhiezer [44], which are more readable, but perhaps somewhat superficial.

The Lagrange problem is solved by using the multiplier rule to generate the necessary additional equations. To this end, a function F is defined as

$$F(\Phi,\Phi',\Xi,\Xi') = G(\Phi,\Phi',\Xi,\Xi') + \Lambda(P\Phi' + Q\Xi') \qquad (7.2.2)$$

where Λ, called the Lagrange multiplier, is a $1 \times r$ matrix; and F is required to satisfy the Euler equations

$$\frac{d}{d\tau}\frac{\partial F}{\partial y_i'} - \frac{\partial F}{\partial y_i} = 0 \qquad (7.2.3)$$

where y_i, $i = 1, \ldots, s + r$, are the elements of Φ and Ξ, i.e.,

$$y_1 = \varphi^1, \ldots, y_s = \varphi^s, y_{s+1} = \xi^1, \ldots, y_{s+r} = \xi^r \qquad (7.2.4)$$

The boundary value problem is now a determined one; it consists of the $2r + s$ equations (7.1.1), (7.2.3) in the $2r + s$ unknowns $\varphi^1, \ldots, \varphi^s$, $\xi^1, \ldots, \xi^r, \lambda^1, \ldots, \lambda^r$ and the $2(r + s)$ boundary conditions (7.1.2).

7.3 When solving a specific Second Problem, one must assign a definite functional form to G in the optimality condition (7.2.1). This assignment is at the discretion of the analyst once a few basic considerations have been taken into account. Among these are the following: the function must be sufficiently "smooth," and it should not yield Euler equations (7.2.3) so degenerate that (7.1.1), (7.2.3) are underdetermined or the boundary conditions (7.1.2) cannot be satisfied.

Since the optimality condition is introduced as an artifice to obtain additional equations, G may be chosen solely for analytical convenience. However, the opportunity exists to optimize the solution in some useful way; for example, one may choose to minimize angular displacements or angular rates, work, energy, or force. (An example of a situation in which it could be advantageous to minimize angular displacements is given in Section 8.4.)

An example will illustrate the possibilities as well as the problems involved in choosing an optimality condition. Since maneuvers of the human body require muscular exertion, it is reasonable to seek those maneuvers which minimize the effort required to perform a given reorientation. Supposing that the extent to which a maneuver is tiring depends on the muscular forces and on the duration of their application, the following optimality condition is proposed:

$$J\ [\Phi,\Xi] = \int_0^1 \sum_i |T_i|\ d\tau \qquad (7.3.1)$$

where T_i is the torque at the i^{th} joint of the body due to the muscular forces exerted, and the summation extends to all the joints of the body. Now, T_i can be found from the inertia forces and torques which act on the component bodies, and, consequently, G depends on the second derivatives of Φ and Ξ, a situation which is not provided for in (7.2.1). However, by introducing a new Euler equation,

$$\frac{d^2}{d\tau^2}\frac{\partial F}{\partial y_i''} - \frac{d}{d\tau}\frac{\partial F}{\partial y_i'} + \frac{\partial F}{\partial y_i} = 0 \qquad (7.3.2)$$

the method may be extended to cover the case at hand. Unfortunately, however, because $|T_i|$ does not have a continuous derivative with respect to any of its arguments at $T_i = 0$, the operations indicated in (7.3.2) cannot be executed, and the optimality condition (7.3.1) must be discarded.

8. Application to a Specific Problem

8.1 In this section the material developed in Section 7 is applied

to the system examined in Section 4, i.e., three bodies in planar motion.† In this context (4.2.7) assumes the role of the constraint equation (7.1.1), and the boundary conditions (7.1.2) are

$$\varphi(0) = \varphi_0 \quad , \quad \varphi(1) = \varphi_n$$

$$\psi(0) = \psi_0 \quad , \quad \psi(1) = \psi_n$$

$$\xi(0) = \xi_0 \quad , \quad \xi(1) = \xi_n \qquad (8.1.1)$$

The following optimality condition is chosen because its use permits a convenient simplification of the resulting equations:

$$J\,[\varphi,\psi,\xi] = \frac{1}{2}\int_0^1 (\varphi'^2 + \psi'^2 + \xi'^2)\,dt \qquad (8.1.2)$$

Differential equations in addition to (4.2.7) are obtained by use of the Euler equation, (7.2.3), together with

$$F\,(\varphi,\varphi',\psi,\psi',\xi') = \frac{1}{2}\,(\varphi'^2 + \psi'^2 + \xi'^2) + \lambda\,(f_1\varphi' + f_2\psi' - \xi') \qquad (8.1.3)$$

which is obtained by reference to (7.2.2). These additional equations are

$$\varphi'' + f_1'\,\lambda + f_1\lambda' - \frac{\partial f_1}{\partial \varphi}\,\varphi'\lambda - \frac{\partial f_2}{\partial \varphi}\,\psi'\lambda = 0 \qquad (8.1.4)$$

$$\psi'' + f_2'\,\lambda + f_2\lambda' - \frac{\partial f_1}{\partial \psi}\,\varphi'\lambda - \frac{\partial f_2}{\partial \psi}\,\psi'\lambda = 0 \qquad (8.1.5)$$

$$\xi' - \lambda = K \qquad (8.1.6)$$

the last of which is an integral of (7.2.3) due to ξ being absent from (8.1.3); and K is the integration constant. The aforementioned

† It may be observed at this point that the system studied in Section 3 is not suitable for this application since, as may be seen by reference to (3.2.11), it is not possible to specify both the initial and final states.

simplification is made by solving (8.1.6) for λ and substituting this expression into (8.1.4) and (8.1.5), thus eliminating λ from the boundary value problem. After making the change of variables

$$\chi = \xi - Kt \tag{8.1.7}$$

and introducing

$$g(\varphi,\psi) = \frac{\partial f_1}{\partial \psi} - \frac{\partial f_2}{\partial \varphi} \tag{8.1.8}$$

the boundary value problem can be compiled from (8.1.4), (8.1.5), (4.2.7), and (8.1.1), respectively:†

$$\varphi'' + f_1 \chi'' + g\psi'\chi' = 0 \tag{8.1.9}$$

$$\psi'' + f_2 \chi'' - g\varphi'\chi' = 0 \tag{8.1.10}$$

$$f_1\varphi' + f_2\psi' - \chi' - K = 0 \tag{8.1.11}$$

$$\varphi(0) = \varphi_0 \quad , \quad \varphi(1) = \varphi_n$$

$$\psi(0) = \psi_0 \quad , \quad \psi(1) = \psi_n$$

$$\chi(0) = \xi_0 \quad , \quad \chi(1) = \chi_n = \xi_n - K \tag{8.1.12}$$

Observe that (8.1.9) - (8.1.11) is a fifth order system of three equations in three unknowns; the system contains a free constant and is subject to the six boundary conditions (8.1.12).

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† In the last of (8.1.12), χ_n is introduced for convenience and is not a boundary value which can be specified (ξ_n still plays this role); this boundary condition might more properly be written $\chi(1) + K = \xi_n$.

8.2 A numerical solution of the boundary value problem (8.1.9) - (8.1.12) is obtained by means of a finite difference technique developed from methods presented by Fox [45]. The essence of this technique is to transform differential equations into transcendental ones by replacing the derivatives which appear in (8.1.9) - (8.1.11) with finite difference formulas. To this end, the unit τ interval is divided into n subintervals separated by $n - 1$ internal pivotal points at which φ, ψ, and χ are unknown, these unknowns being denoted φ_i, ψ_i, and χ_i, $i = 1, \ldots, n - 1$. One other unknown, χ_n, remains after K is replaced by $\xi_n - \chi_n$. Using central difference formulas, (8.1.9) - (8.1.11) are written as finite difference equations at each of the internal pivotal points, and (8.1.11) is written at the final point by using an approximate backward difference formula. The resulting $3n - 2$ transcendental equations may be solved for the $3n - 2$ unknown pivotal values.

The central difference formulas, where y_i may represent φ_i, ψ_i, or χ_i, are given by Fox [46]:

$$y_i' = \frac{n}{2} (y_{i+1} - y_{i-1}) + nC_1 y_i \tag{8.2.1}$$

where

$$C_1 = - \frac{1}{6} \mu\delta^3 + \frac{1}{30} \mu\delta^5 - \ldots \tag{8.2.2}$$

and

$$y_i'' = n^2 (y_{i+1} - 2 y_i + y_{i-1}) + n^2 C_2 y_i \tag{8.2.3}$$

where

$$C_2 = - \frac{1}{12} \delta^4 + \frac{1}{90} \delta^6 - \ldots \tag{8.2.4}$$

In these formulas, C_1, C_2, δ, and μ are understood to operate on y_i, and the meaning of δ (the central difference operator) and μ (the averaging operator) may be obtained from examples:

$$\delta^2 y_i = \delta y_{i+1/2} - \delta y_{i-1/2} = y_{i+1} - y_i - y_i + y_{i-1} \qquad (8.2.5)$$

$$\mu\delta^3 y_i = \frac{1}{2}\left(\delta^3 y_{i+1/2} + \delta^3 y_{i-1/2}\right) \qquad (8.2.6)$$

The approximate backward difference formula is an eighth-order form given by Bickley [47]:

$$y_i' = \frac{n}{40320}\left(5040\, y_{i-8} - 46080\, y_{i-7} + 188160\, y_{i-6} - 451584\, y_{i-5} + 705600\, y_{i-4} - 752640\, y_{i-3} + 564480\, y_{i-2} - 322560\, y_{i-1} + 109584\, y_i\right) \qquad (8.2.7)$$

The system of transcendental equations is

$$\varphi_{i+1} - 2\varphi_i + \varphi_{i-1} + C_2\varphi_i + f_1(\chi_{i+1} - 2\chi_i + \chi_{i-1}) + f_1 C_2\, \chi_i + \frac{g}{4}(\psi_{i+1} - \psi_{i-1})(\chi_{i+1} - \chi_{i-1}) + \frac{g}{2}\left[(\psi_{i+1} - \psi_{i-1})\, C_1\, \chi_i + (\chi_{i+1} - \chi_{i-1})\, C_1 \psi_i\right] + g C_1 \psi_i C_1\, \chi_i = 0 \qquad (8.2.8)$$

$$\psi_{i+1} - 2\psi_i + \psi_{i-1} + C_2\psi_i + f_2(\chi_{i+1} - 2\chi_i + \chi_{i-1}) + f_2 C_2 \chi_i - \frac{g}{4}(\varphi_{i+1} - \varphi_{i-1})(\chi_{i+1} - \chi_{i-1}) - \frac{g}{2}\left[(\varphi_{i+1} - \varphi_{i-1})\, C_1\, \chi_i + (\chi_{i+1} - \chi_{i-1})\, C_1\, \varphi_i\right] - g C_1 \varphi_i C_1\, \chi_i = 0 \qquad (8.2.9)$$

$$f_1(\varphi_{i+1} - \varphi_{i-1}) + 2f_1 C_1 \varphi_i + f_2(\psi_{i+1} - \psi_{i-1}) + 2f_2 C_1 \psi_i - \chi_{i+1} + \chi_{i-1} - 2C_1\, \chi_i + \frac{2}{n}\chi_n - \frac{2}{n}\xi_n = 0 \qquad (8.2.10)$$

$$\frac{f_1}{20160}(5040\,\varphi_{n-8} - 46080\,\varphi_{n-7} + 188160\,\varphi_{n-6} - 451584\,\varphi_{n-5} + 705600\,\varphi_{n-4}$$

$$- 752640\,\varphi_{n-3} + 564480\,\varphi_{n-2} - 322560\,\varphi_{n-1} + 109584\,\varphi_n) + \frac{f_2}{20160}(5040\,\psi_{n-8}$$

$$- 46080\,\psi_{n-7} + 188160\,\psi_{n-6} - 451584\,\psi_{n-5} + 705600\,\psi_{n-4} - 752640\,\psi_{n-3}$$

$$+ 564480\,\psi_{n-2} - 322560\,\psi_{n-1} + 109584\,\psi_n) - \frac{5040}{20160}\chi_{n-8} + \frac{46080}{20160}\chi_{n-7}$$

$$- \frac{188160}{20160}\chi_{n-6} + \frac{451584}{20160}\chi_{n-5} - \frac{705600}{20160}\chi_{n-4} + \frac{752640}{20160}\chi_{n-3} - \frac{564480}{20160}\chi_{n-2}$$

$$+ \frac{322560}{20160}\chi_{n-1} - \frac{109584}{20160}\chi_n + \frac{2}{n}\chi_n - \frac{2}{n}\xi_n = 0 \qquad (8.2.11)$$

where $i = 1, \ldots, n-1$, and f_1, f_2, and g are evaluated at the i^{th} pivotal point in (8.2.8) - (8.2.10) and at the n^{th} pivotal point in (8.2.11).

The transcendental system (8.2.8) - (8.2.11) is solved by a method due to Kane [48] which is well suited to use with a digital computer. The equations are prepared for numerical solution by considering the unknown pivotal values to be functions of a parameter z and by replacing the right-hand sides of (8.2.8) - (8.2.11) with $k_{3i-2}(1-z)$, $k_{3i-1}(1-z)$, $k_{3i}(1-z)$, and $k_{3n-2}(1-z)$, respectively, where k_{3i-2}, etc., are free constants. For example, (8.2.10) becomes

$$f_1(\varphi_{i+1} - \varphi_{i-1}) + 2f_1C_1\varphi_i + f_2(\psi_{i+1} - \psi_{i-1}) + 2f_2C_1\psi_i$$

$$- \chi_{i+1} + \chi_{i-1} - 2C_1\chi_i + \frac{2}{n}\chi_n - \frac{2}{n}\xi_n = k_{3i}\,(1 - z) \qquad (8.2.12)$$

Next, these equations are differentiated with respect to z, such differentiations being denoted by asterisks; if (8.2.10) is again used as an example, (8.2.12) becomes

$$\frac{\partial f_1}{\partial \varphi}(\varphi_{i+1} - \varphi_{i-1})\,\varphi_i^* + \frac{\partial f_1}{\partial \psi}(\varphi_{i+1} - \varphi_{i-1})\,\psi_i^* + f_1(\varphi_{i+1}^* - \varphi_{i-1}^*)$$

$$+ 2\frac{\partial f_1}{\partial \varphi} C_1 \varphi_i \varphi_i^* + 2\frac{\partial f_1}{\partial \psi} C_1 \varphi_i \psi_i^* + \frac{\partial f_2}{\partial \varphi}(\psi_{i+1} - \psi_{i-1})\,\varphi_i^*$$

$$+ \frac{\partial f_2}{\partial \psi}(\psi_{i+1} - \psi_{i-1})\,\psi_i^* + f_2(\psi_{i+1}^* - \psi_{i-1}^*) + 2\frac{\partial f_2}{\partial \varphi} C_1 \psi_i \varphi_i^*$$

$$+ 2\frac{\partial f_2}{\partial \psi} C_1 \psi_i \psi_i^* - \chi_{i+1}^* + \chi_{i-1}^* + \frac{2}{n}\chi_n^* = -k_{3i} \qquad (8.2.13)$$

This equation, together with corresponding equations obtained from (8.2.8), (8.2.9), and (8.2.11), forms a system of first order differential equations which, because the quantities marked with asterisks appear linearly, may, by inversion, be written in a form suitable for numerical integration. By integrating from $z = 0$ to $z = 1$, a solution of (8.2.8) - (8.2.11) may be obtained at $z = 1$, since the quantities introduced on the right-hand sides of (8.2.8) - (8.2.11) vanish when $z = 1$. The initial values required for the integration are obtained by making a guess at the solution of the boundary value problem (8.1.9) - (8.1.12), and the values of k_{3i-2}, etc., associated with these initial values are obtained from (8.2.12) and its corresponding equations.

Solution of the transcendental equations (8.2.8) - (8.2.11) is complicated by the fact that the finite difference corrections $C_1\varphi_i$, etc., are not known in advance and cannot be expressed in terms of the unknown pivotal values. This difficulty is overcome by an iterative procedure in which values are assigned to the finite difference corrections, the transcendental equations are solved for the pivotal values, and this solution is used to redetermine the difference corrections. Since the finite difference corrections are assumed small, they are initially set

to zero; thereafter they are computed by extrapolating the solution of the boundary value problem outside the unit τ interval, computing higher differences, and determining the difference corrections from (8.2.2) and (8.2.4). The iterative solution of the transcendental equations continues until the difference between successive solutions becomes small. It should be noted in passing that, on all iterations after the first, the solution from the previous iteration provides a good set of initial values for the numerical integration.

In the actual implementation of the finite difference technique, several questions arise, and, consequently, the remainder of this subsection is devoted to a discussion of some of these questions. To begin, a value of n must be chosen. This value must be no smaller than the order of the approximate backward difference formula used--eight in this case, and the author recommends $n = 5$ as an absolute lower limit for acceptable results in view of the character of the solution. Large values of n increase the amount of computation approximately as n^3 due to the numerical inversion required at each step of integration when solving the transcendental equations.

A related question is that of the order to be used in the difference formulas and difference corrections. It seems advisable to use orders as large as practicable for three reasons: higher order formulas yield more accurate results (an assumption); the use of a high order formula in (8.2.11) reduces "oscillation" of the solution and improves accuracy according to Fox [49]; although high order formulas complicate the equations considerably, they result in very little extra computation. However, the assumption noted above is not universally valid; cases have

been found in which derivatives can be computed more accurately by a low order formula of the form of (8.2.7) than by a high order formula of the same form.

Finally, a suitable method must be found for extrapolating the solution when computing difference corrections. Of several methods tried, the one which seems to work best involves using the pivotal values to compute $\varphi'_{n/2}$, $\psi'_{n/2}$, and $\chi'_{n/2}$ (assuming n to be even) by means of approximate central difference formulas (see Bickley [47]). By using these derivatives and the pivotal values $\varphi_{n/2}$, $\psi_{n/2}$, and $\chi_{n/2}$ as "initial" conditions, (8.1.9) - (8.1.11) may be integrated numerically, both backward and forward from $\tau = 1/2$, to give the desired extrapolation. On the unit τ interval, values resulting from the numerical integration, i.e., $\varphi(1/n)$, etc., are used rather than pivotal values, i.e., φ_1, etc., for computing the difference corrections.

8.3 The finite difference technique possesses three limitations: it is inherently inaccurate, the accuracy of the solution cannot be determined, and the solution is given in terms of a small number of pivotal values. These difficulties may be overcome by switching to another technique of solution once an approximate solution has been obtained by the finite difference technique. In the improvement technique developed here, the boundary value problem (8.1.9) - (8.1.12) is solved iteratively by considering it as an initial value problem, iteration being used to find a set of initial derivatives which, when used with the initial values of (8.1.12), will yield the final values of (8.1.12). A set of initial derivatives for the first iteration is obtained from the finite difference solution by numerical differentiation [47].

The statement of the initial value problem is similar to (8.1.9) - (8.1.12), except that (8.1.11) must be differentiated in order to have a system amenable to numerical integration:

$$\varphi'' + f_1 \chi'' + g\psi'\chi' = 0 \tag{8.3.1}$$

$$\psi'' + f_2\chi'' - g\varphi'\chi' = 0 \tag{8.3.2}$$

$$f_1\varphi'' + f_2\psi'' - \chi'' + \frac{\partial f_1}{\partial \varphi}\varphi'^2 + \left(\frac{\partial f_1}{\partial \psi} + \frac{\partial f_2}{\partial \varphi}\right)\varphi'\psi' + \frac{\partial f_2}{\partial \psi}\psi'^2 = 0 \tag{8.3.3}$$

$$\varphi(0) = \varphi_0 \quad , \quad \varphi'(0) = \varphi_0'$$

$$\psi(0) = \psi_0 \quad , \quad \psi'(0) = \psi_0'$$

$$\chi(0) = \xi_0 \quad , \quad \chi'(0) = \chi_0' \tag{8.3.4}$$

Two matrices are introduced to facilitate the work:

$$\Delta^j = \begin{bmatrix} \varphi_0^{j\prime} - \varphi_0^{j-1\prime} \\ \psi_0^{j\prime} - \psi_0^{j-1\prime} \\ \chi_0^{j\prime} - \chi_0^{j-1\prime} \end{bmatrix} \tag{8.3.5}$$

where j is the iterative index, and

$$D^j = \begin{bmatrix} \frac{\partial\varphi(1)}{\partial\varphi_0'} & \frac{\partial\varphi(1)}{\partial\psi_0'} & \frac{\partial\varphi(1)}{\partial\chi_0'} \\ \frac{\partial\psi(1)}{\partial\varphi_0'} & \frac{\partial\psi(1)}{\partial\psi_0'} & \frac{\partial\psi(1)}{\partial\chi_0'} \\ \left\{\frac{\partial\chi(1)}{\partial\varphi_0'} + f_1\right\} & \left\{\frac{\partial\chi(1)}{\partial\psi_0'} + f_2\right\} & \left\{\frac{\partial\chi(1)}{\partial\chi_0'} - 1\right\} \end{bmatrix} \tag{8.3.6}$$

where $\frac{\partial\varphi(1)}{\partial\varphi_0'}$, e.g., is understood to be the ratio of the change in $\varphi(1)$

to the change in φ_0' when the numerical integration of (8.3.1) - (8.3.4) is performed twice, using $\varphi_0^{j-1'}$ and a slightly differing value. (Reasonable values for the increments in $\varphi_0^{0'}$, $\psi_0^{0'}$, and $\chi_0^{0'}$ may be found by subtracting values computed from different order forward difference formulas; for $j \geq 2$, the elements of Δ^{j-1} provide good increments.) The additional terms in the third row of (8.3.6), which appear because K changes value from one iteration to the next, are evaluated using φ_0 and ψ_0.

For each iteration, new initial derivatives are predicted by a method very similar to the Newton-Raphson method [50]. Specifically, the corrections, Δ^j, to the initial derivatives are solutions of

$$D^j \Delta^j = \begin{bmatrix} \varphi(1) - \varphi_n \\ \psi(1) - \psi_n \\ \chi(1) - \xi_n + K \end{bmatrix} \qquad (8.3.7)$$

where, by reference to (8.1.11),

$$K = f_1(\varphi_0, \psi_0)\ \varphi_0^{j-1'} + f_2(\varphi_0, \psi_0)\ \psi_0^{j-1'} - \chi_0^{j-1'} \qquad (8.3.8)$$

and $\varphi(1)$, $\psi(1)$, and $\chi(1)$, obtained when D^j is computed, correspond to $\varphi_0^{j-1'}$, $\psi_0^{j-1'}$, $\chi_0^{j-1'}$. To summarize the iterative process, D^j and the right-hand side of (8.3.7) are computed, Δ^j is then obtained from (8.3.7), and the initial derivatives are corrected, i.e.,

$$\begin{bmatrix} \varphi_0^{j'} \\ \psi_0^{j'} \\ \chi_0^{j'} \end{bmatrix} = \begin{bmatrix} \varphi_0^{j-1'} \\ \psi_0^{j-1'} \\ \chi_0^{j-1'} \end{bmatrix} + \Delta^j \qquad (8.3.9)$$

Iteration continues until the final values, $\varphi(1)$, $\psi(1)$, and $\xi(1)$ $(= \chi(1) + K)$, become sufficiently accurate.

8.4 The material developed in this section is applied here to the three body human described in Section 4.6. The constants used are those for the Air Force mean man given in Section 4.6, but the boundary values differ since the motion now begins and ends with the body in a "standing" position with respect to the internal variables. When one seeks a reorientation of five degrees, the boundary values are

$$\varphi_0 = 180^\circ \quad , \quad \varphi_n = 180^\circ$$
$$\psi_0 = 0 \quad , \quad \psi_n = 0$$
$$\xi_0 = 0 \quad , \quad \xi_n = 5^\circ$$

A plot of the solution of this Second Problem, obtained on a Burroughs B5500 computer using the program listed in Appendix 4, is shown in Fig. 8.4. Notice that φ drops almost to 60 degrees, a position that is difficult, but not impossible (see Section 4.6), for a man to attain. Solutions of this problem for larger reorientations, namely 15° and 30°, have also been obtained; the shape of the solution curves is about the same for these cases except that the ordinates (changes in the angles) are magnified. Even though the reorientations obtained are more impressive, these solutions are not presented here because φ becomes too small to be physically possible for a human being.

Some details of the solution process are given here to serve as examples of some of the topics treated generally in Sections 8.2 and 8.3. First, the initial guess of the pivotal values, which is necessary in order to solve the transcendental equations, is made so as to "suggest"

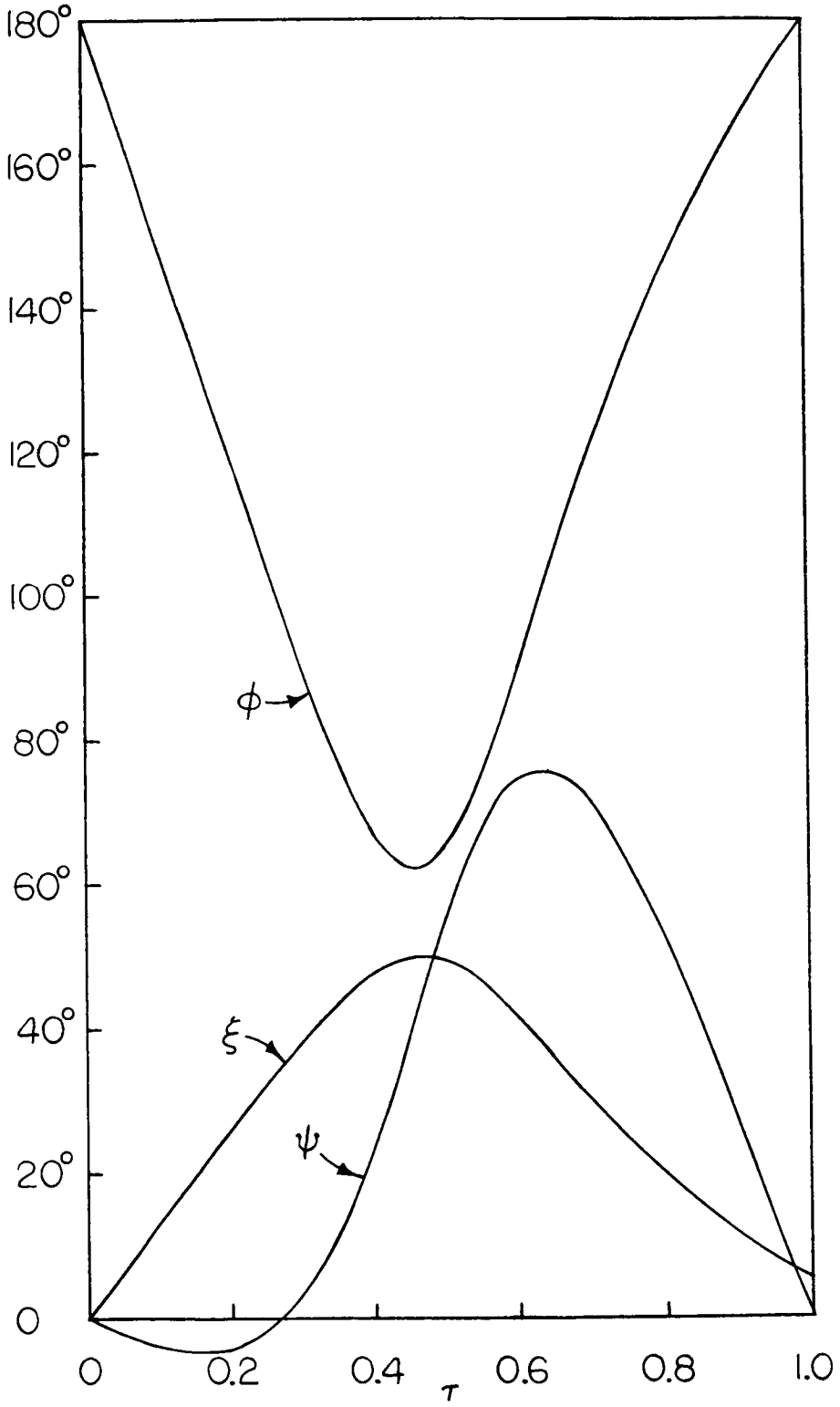

Fig. 8.4

to the computer values which are physically attainable for a man. The guess for φ proceeds linearly from 3 (radians) at $\tau = 0$ to 1.5 at $\tau = 1/2$ and returns linearly to 3 at $\tau = 1$; similarly, ψ proceeds linearly from 0 to 1.5 and returns to 0. For lack of any better information, χ is guessed to be identically zero. By reference to Fig. 8.4, one can see that the guesses for φ and ψ are quite reasonable; however, since the computer solution gives $K = 114$, the guess for χ, in view of (8.1.7), is very poor.

In addition to the input data already mentioned, several other quantities must be specified in order to control the solution process; data used for the problem at hand are given at the end of the program listing in Appendix 4. When a value of eight is used for n, the solution shown in Fig. 8.4 is obtained in two iterations of the finite difference technique and three iterations of the improvement technique; the processor time required on the B5500 is 1.5 minutes, about half of which is spent on the first finite difference iteration.

9. A Variant of the Second Problem

9.1 The Second Problem is defined in Section 2.2 as one in which it is desired to know how to prescribe the internal variables so as to achieve a given reorientation (of the main body). Suppose, instead, that it is desired to know how to prescribe the internal variables so as to achieve a given motion of the main body. This problem is akin to the Second Problem in the sense that it requires the determination of the cause that will produce a given effect. On the other hand, it is mathematically similar to the First Problem since it may be solved by specifying the external variables as functions of time and integrating the

equations of motion as an initial value problem in order to determine how the internal variables must be prescribed.

By reference to (7.1.1), equations of motion for systems of the type under consideration may be written

$$P(\Phi)\ \Phi' + \Omega = 0 \qquad (9.1.1)$$

where Ω, a column matrix composed of the angular rates of the main body multiplied by the duration of the maneuver T (see footnote, p.15), replaces $Q(\Xi)\ \Xi'$, and all other terms are defined as in Section 7.1. To solve this variant of the Second Problem, Ω is specified as a function of τ, (9.1.1) is solved to give Φ' explicitly, and the resulting expression is integrated to give $\Phi(\tau)$. However, it is possible to obtain an explicit expression for Φ' only if P is nonsingular along the path of motion. Consequently, internal variables may be prescribed as functions of τ so as to achieve a specified motion of the main body only if $\det P\ [\Phi(\tau)] \neq 0$, $0 \leq \tau \leq 1$. To illustrate this, consider a system composed of two rectangular parallelepipeds connected at their corners by a ball and socket joint. Using equations of motion (5.2.9) - (5.2.11), an attempt was made to find the internal variables corresponding to eight specified motions of the main body, all of them rotations of less than 90° about an axis fixed in the main body and fixed in an inertial reference frame. (Euler parameters, Appendix 2, were used to specify these motions.) For the eight motions specified, a corresponding Φ existed in only one case.

9.2 The system examined in Section 3.3 provides a simple example of the fact that it may be possible to specify the reorientation, but

impossible to specify the motion, of the main body. Considering a system composed of rods of length ratio 3/10, suppose that it is desired to find $\varphi(\tau)$ such that the main body will move monotonically from $\xi = 0$ to $\xi = 5^\circ$. It may be seen from the curve for $\frac{a_1}{a_0} = \frac{3}{10}$ in Fig. 3.3a that such a φ does not exist since ξ cannot increase monotonically beyond 3.5 degrees. However, a reorientation of 5 degrees may be achieved by changing φ from 0 to -167 degrees, as may be seen by reflecting the curve for $\frac{a_1}{a_0} = \frac{3}{10}$ through the origin.

When $\frac{d\xi}{d\varphi} = 0$, (3.2.11) gives rise to

$$\varphi = 2 \tan^{-1} \sqrt{\frac{1 + a}{1 - a}} \qquad (9.2.1)$$

where a is defined in (3.3.1); $a = -1$ when $\frac{a_1}{a_0}$ is approximately 0.65. Hence, when the length ratio is less than 0.65, the maximum monotonic rotation of the main body is limited, and the limit may be obtained by using (9.2.1) for $\varphi(1)$ in (3.2.11).

IV. CONCLUSIONS

Although detailed conclusions are presented in the preceding material, it may be helpful to summarize these briefly.

Investigators in several fields have studied motions of the human body in light of the principles of mechanics. But, as shown in Section 1, some of this work lacks analytical rigor and the rest tends to be incompletely developed; i.e., correct equations of motion may be presented as an end in themselves, rather than as a means to a well-defined end.

The need for a sound analytical approach to the problem of human motion is pointed up in Sections 3 and 5, which illustrate some of the subtleties of this problem. For example, even for the simplest case considered, that of planar motion of two bodies, it is found that both the governing equations and the motion itself are rather complicated, mainly because the mass center of the system does not remain fixed in either of the bodies. And it is surprising to find that both bodies of this simple model can rotate in the same sense even when the angular momentum of the system is zero, a conclusion that could scarcely be reached by simple "action and reaction" arguments. Moving on to the general motion of two bodies (Section 5), one finds that the equations become considerably more complicated and that there is now no hope of a closed form solution. In addition, it appears that very small changes in the internal variables can produce large out-of-plane rotations of the main body.

Sections 4 and 6 deal with the subject of reorienting the human body by moving the limbs. Although it may be obvious that the body can be reoriented by means of continuous, rotary limb motions, it is not clear that reorientations can also be obtained by swinging the limbs into some position and then returning them to the original position without performing a net rotation. Section 4 shows that this is, indeed, possible. Quantitative information is given in Section 6 for a maneuver having potential applicability to "space walking." This maneuver, in which a rotation about the spine axis is produced by swinging one limb to the side and one limb forward, is unnatural in the sense that one would not expect it to produce rotation about the spine axis, and it would probably not have been discovered by experimentation or by superficial analytical methods. One of these maneuvers, in which the subject swings one leg forward and one leg to the side, yields a 20 degree rotation about the spine axis; a 180 degree rotation would require nine performances of the maneuver and would take about one minute if an astronaut can move at one half the speed of a man on earth who is unencumbered by a space suit.

The general problem of relating reorientation of the body to relative motion of body parts may be viewed in two ways, as described in Section 2. The so-called First Problem deals with specified relative motions of the body parts and is easily solved as an initial value problem. The Second Problem, on the other hand, is mathematically complex, and may possess either no solution or several solutions. A general method of attack is described in Section 7, and the practical feasibility of this method is demonstrated in Section 8 in connection with a specific example.

APPENDIX 1

Two-Body Model of a Cat

One explanation of the righting maneuver of a freely falling cat may be formulated in terms of a dynamical model which readily admits of analysis. According to this explanation, the cat first bends its spine in the middle, this bend being maintained throughout the maneuver. Next, both the fore and hind parts of the body rotate about their respective spine axes at the same rate relative to a reference frame in which the spine axes are fixed; an appropriate counterrotation of this reference frame takes place; and the combination of this rotation and those about the spine axes leads to the turning-over of the cat.†

It is assumed that the fore and hind parts of the body may be adequately represented by identical axisymmetric bodies, denoted B_1 and B_2, and that the masses of all other parts are negligible. Bodies B_1 and B_2 rotate at a rate ω relative to a reference frame in which the spine axes and the subscripted unit vectors in Fig. A-1a are fixed, and this reference frame rotates at a rate Ω about an axis parallel to the inertially fixed unit vector $\underline{i}$. Each body has longitudinal and transverse moments of inertia I and J, respectively, for its mass center, either C_1 or C_2.

The angular momenta of the bodies relative to their respective mass centers are

† This explanation was advanced by Rademaker and Ter Braak [32], who also presented an analysis essentially the same as the one given here. This material is presented because it appears to be relatively unknown and the reference is inaccessible to many engineering readers.

$$\underline{A}^{B_1/C_1} = I(\omega + \Omega \cos \alpha)\, \underline{i}_1 + J\, \Omega \sin \alpha \underline{j}_1 \qquad \text{(A-1.1)}$$

and

$$\underline{A}^{B_2/C_2} = I(\omega + \Omega \cos \alpha)\, \underline{i}_2 - J\, \Omega \sin \alpha \underline{j}_2 \qquad \text{(A-1.2)}$$

The sum of these is the angular momentum of the entire system relative to its mass center, C*:

$$\underline{A}^{S/C^*} = 2[I(\omega + \Omega \cos \alpha) \cos \alpha + J\, \Omega \sin^2 \alpha]\, \underline{i} \qquad \text{(A-1.3)}$$

If the system starts from rest, a relationship between ω and Ω may be obtained by equating $\underline{A}^{S/C^*}$ to zero, which leads to

$$\frac{\omega}{\Omega} = \frac{(1 - I/J) \cos^2 \alpha - 1}{(I/J) \cos \alpha} \qquad \text{(A-1.4)}$$

The ratio ω/Ω must satisfy a second, purely kinematic relationship if the cat is to land on its feet after being released in an upside down position; i.e., the angle through which each of the bodies B_1 and B_2 turns about its axis relative to the plane determined by these axes must be properly related to the angle through which this plane itself turns.

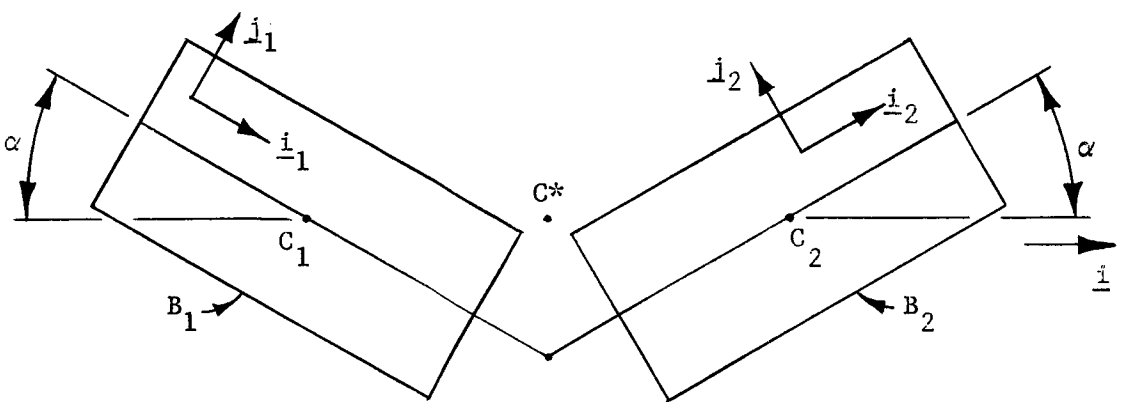

Fig. A-1a

If the spine is bent such that the middle of the back is the lowest point of the cat when released and the highest point when landing, then the simplest case is that in which B_1 and B_2 each perform one full revolution while the plane determined by their axes rotates through 180 degrees. This occurs if $\omega/\Omega = -2$, and use of this value in (A-1.4) leads to the relationship between α and I/J displayed in Fig. A-1b.

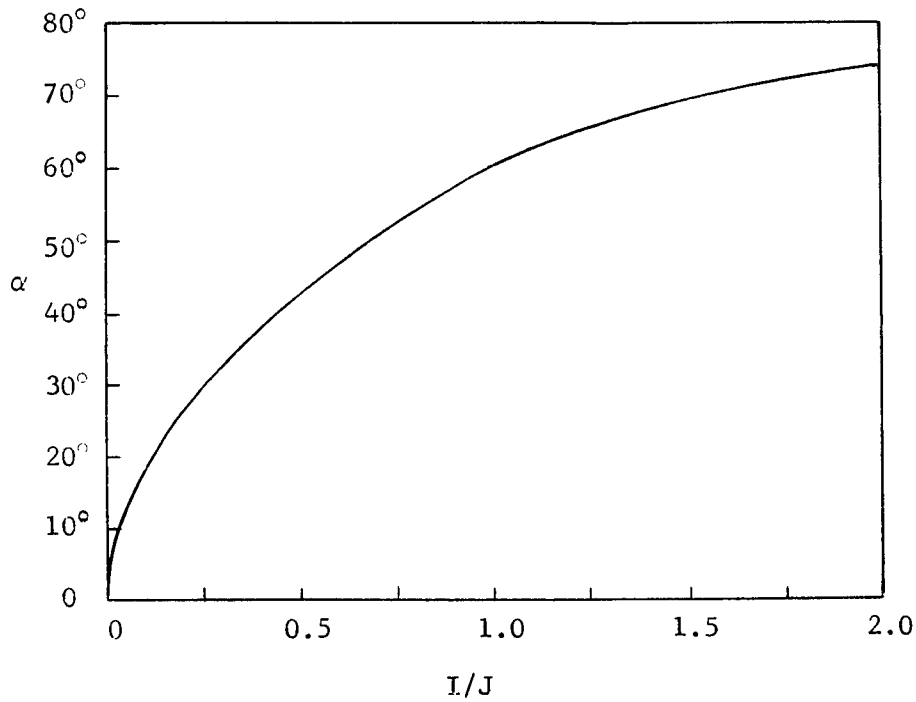

Fig. A-1b

APPENDIX 2

Kinematical Relationships

The specification of the orientation of a rigid body relative to a reference frame may be accomplished by using one of a number of schemes, each with its own advantages and peculiarities. This appendix presents several of these schemes briefly.

Two-Axis Euler Angles

If R is a reference frame in which right-handed, mutually perpendicular axes X, Y, Z are fixed, any orientation of R relative to a reference frame R_0 in which right-handed, mutually perpendicular axes X_0, Y_0, Z_0 are fixed may be described by specifying three angles, φ_1, φ_2, and φ_3, called "two-axis" Euler angles, and defined as follows: align X, Y, Z with X_0, Y_0, Z_0 and then perform successive right-handed rotations of R of amount φ_1 about X, φ_2 about Y, and φ_3 about X. It is the repeated use of the X-axis that suggests the name two-axis Euler angles.

When working with vectorial quantities expressed in terms of two sets of unit vectors, one set, $\underline{i}$, $\underline{j}$, $\underline{k}$, which remain respectively parallel to X, Y, Z and one set, $\underline{i}_0$, $\underline{j}_0$, $\underline{k}_0$, which remain respectively parallel to X_0, Y_0, Z_0, it is often necessary to transofrm the vectorial quantities from one set of unit vectors to the other. The transformation formulas may be presented as follows:

	$\underline{i}_0$	$\underline{j}_0$	$\underline{k}_0$
$\underline{i}$	c_2	s_2s_1	$-s_2c_1$
$\underline{j}$	s_2s_3	$c_3c_1 - c_2s_3s_1$	$c_3s_1 + c_2s_3c_1$
$\underline{k}$	s_2c_3	$-s_3c_1 - c_2c_3s_1$	$-s_3s_1 + c_2c_3c_1$

(A-2.1)

Here, as before, the subscripted letters s and c denote trigonometric functions, i.e., $c_i = \cos \varphi_i$ and $s_i = \sin \varphi_i$. The transformation table (A-2.1) may be read either across or down, e.g., $\underline{i}_0 = c_2\underline{i} + s_2s_3\underline{j} + s_2c_3\underline{k}$ and $\underline{k} = s_2c_3\underline{i}_0 - (s_3c_1 + c_2c_3s_1)\,\underline{j}_0 - (s_3s_1 - c_2c_3c_1)\,\underline{k}_0$.

Three differential equations relate the two-axis Euler angles to ω_1, ω_2, ω_3, the X, Y, Z measure numbers of the angular velocity of R in R_0:

$$\dot{\varphi}_1 = (\omega_2 s_3 + \omega_3 c_3)/s_2$$

$$\dot{\varphi}_2 = \omega_2 c_3 - \omega_3 s_3$$

$$\dot{\varphi}_3 = \omega_1 - (\omega_2 s_3 + \omega_3 c_3)\,(c_2/s_2) \qquad \text{(A-2.2)}$$

Note that (A-2.2) are singular when $\sin \varphi_2 = 0$.

Two-axis Euler angles are most useful in problems where the rotations φ_1, φ_2, and φ_3 have some physical significance; for example, when related to a spinning top, φ_1 is the precession angle, φ_2 measures the inclination of the spin axis, and φ_3 is the spin angle. It should be noted that there is an ambiguity in the two-axis description when $\sin \varphi_2 = 0$, for then φ_1 and φ_3 are indistinguishable. In addition to the singularity in (A-2.2), this implies that small values of φ_1 and φ_3 are not assured for small misalignments of X, Y, Z and X_0, Y_0, Z_0.

Three-Axis Euler Angles

Three-axis Euler angles θ_1, θ_2, θ_3 are generated in the same manner as the two-axis variety, except that the third rotation is performed about Z rather than about X. The unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ and $\underline{i}_0$, $\underline{j}_0$, $\underline{k}_0$, defined as before, are now related as follows:

	$\underline{i}_0$	$\underline{j}_0$	$\underline{k}_0$
$\underline{i}$	c_2c_3	$s_1s_2c_3 + c_1s_3$	$-c_1s_2c_3 + s_1s_3$
$\underline{j}$	$-c_2s_3$	$-s_1s_2s_3 + c_1c_3$	$c_1s_2s_3 + s_1c_3$
$\underline{k}$	s_2	$-s_1c_2$	c_1c_2

(A-2.3)

where now $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$. And (A-2.2) are replaced with

$$\dot{\theta}_1 = (\omega_1c_3 - \omega_2s_3)/c_2$$
$$\dot{\theta}_2 = \omega_1s_3 + \omega_2c_3$$
$$\dot{\theta}_3 = \omega_3 - (\omega_1c_3 - \omega_2s_3)(s_2/c_2) \qquad \text{(A-2.4)}$$

Three-axis Euler angles are useful in problems where X, Y, Z and X_0, Y_0, Z_0 are normally closely aligned. For example, it is often desirable for the orientation of a spacecraft to remain essentially fixed in some reference frame. When the misalignment of X, Y, Z and X_0, Y_0, Z_0 is small, θ_1, θ_2, and θ_3 are also small, and visualization of the orientation may be facilitated by making the three rotations, in any order, about either the X, Y, Z axes, or the X_0, Y_0, Z_0 axes. That is, for small θ_1, θ_2, and θ_3, the order of the rotations and the exact direction of the axes about which the rotations are made are relatively unimportant.

Corresponding to the ambiguity in the two-axis Euler angles, there is an ambiguity in the three-axis Euler angle description when $\cos\theta_2 = 0$, and this is reflected in the singularity in (A-2.4) at these values of θ_2.

Euler Parameters

Euler parameters stem from a theorem due to Euler which states that any change in the orientation of a rigid body relative to a reference frame may be obtained by performing a single rotation about an axis which is fixed both in the body and in the reference frame. Consider two reference frames, R and R_0, and let $\underline{i}$, $\underline{j}$, $\underline{k}$ and $\underline{i}_0$, $\underline{j}_0$, $\underline{k}_0$ be two right-handed sets of mutually perpendicular unit vectors, fixed in R and in R_0, respectively. If these two sets of unit vectors are initially coincident, and if a rotation of R relative to R_0 is specified in terms of θ, the angle of rotation of R relative to R_0, and $\underline{n}$, a unit vector parallel to the axis of rotation and with sense such that increasing θ is a right-handed rotation of R about a line parallel to $\underline{n}$, then, by defining a vector $\underline{\varepsilon}$ as

$$\underline{\varepsilon} = \sqrt{2} \sin \frac{\theta}{2} \underline{n} \qquad \text{(A-2.5)}$$

the Euler parameters ε_1, ε_2, ε_3, and η are obtained from

$$\eta = \sqrt{2} \cos \frac{\theta}{2} \qquad \text{(A-2.6)}$$

and

$$\begin{aligned} \underline{\varepsilon} &= \varepsilon_1 \underline{i} + \varepsilon_2 \underline{j} + \varepsilon_3 \underline{k} \\ &= \varepsilon_1 \underline{i}_0 + \varepsilon_2 \underline{j}_0 + \varepsilon_3 \underline{k}_0 \end{aligned} \qquad \text{(A-2.7)}$$

Since only three quantities are required to determine the orientation of R relative to R_0, a constraint equation relates ε_1, ε_2, ε_3, and η; i.e., it follows from (A-2.5) - (A-2.7) that

$$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \eta^2 = 2 \qquad \text{(A-2.8)}$$

Unit vectors fixed in R and R_0 are related by the transformation

	$\underline{i}_0$	$\underline{j}_0$	$\underline{k}_0$
$\underline{i}$	$1 - \epsilon_2^2 - \epsilon_3^2$	$\epsilon_1\epsilon_2 + \epsilon_3\eta$	$\epsilon_1\epsilon_3 - \epsilon_2\eta$
$\underline{j}$	$\epsilon_2\epsilon_1 - \epsilon_3\eta$	$1 - \epsilon_3^2 - \epsilon_1^2$	$\epsilon_2\epsilon_3 + \epsilon_1\eta$
$\underline{k}$	$\epsilon_3\epsilon_1 + \epsilon_2\eta$	$\epsilon_3\epsilon_2 - \epsilon_1\eta$	$1 - \epsilon_1^2 - \epsilon_2^2$

(A-2.9)

and the differential equations relating Euler parameters to angular velocity can be expressed as

$$\begin{bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \\ \dot{\eta} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \eta \end{bmatrix} \qquad \text{(A-2.10)}$$

where ω_1, ω_2, ω_3 are the $\underline{i}$, $\underline{j}$, $\underline{k}$ measure numbers of the angular velocity of R in R_0.

On the surface, Euler parameters appear to be ideal for visualizing changes in orientation since only a single rotation about a fixed axis is required; but in specific instances it is not clear that Euler parameters are more useful for this purpose than Euler angles. As an example, try to visualize the change in orientation of one's own body resulting from a rotation of 120 degrees about an axis passing through the right shoulder and the left hip. Compare this with the same change in orientation given in terms of successive rotations about the spine, eye, and

shoulder axes. Since the latter rotations are relatively easy to visualize individually because of the axes involved, three of these rotations performed successively may be easier to visualize than a single rotation about a skew axis. For small changes in orientation, the first three Euler parameters are approximately $1/\sqrt{2}$ times the corresponding three-axis Euler angles, but the Euler angles are probably easier to work with in such situations.

Numerical integration of (A-2.9) is relatively fast because these equations do not contain trigonometric functions; but this advantage is of relatively little importance for the integrations done in connection with this dissertation since the equations of motion are quite complex in themselves.

Although there is no ambiguity in the Euler parameter description as there is for Euler angles, there is an ambiguity in the orientation described by the Euler parameters since they are periodic functions of θ. However, the ambiguity of orientation is of little concern in this work since changes in orientation of less than one revolution are usually sufficient.

APPENDIX 3

The computer program (written in Extended Algol for the Burroughs B5500) used with Section 6, together with the data corresponding to the point $\varphi^* = 180^\circ$, $\psi^* = 30^\circ$ in Fig. 6.3.

```
BEGIN COMMENT: THREE BODIES IN GENERAL MOTION. SEE SECTION 6;
COMMENT : NEEDS KUTTAMERSON, GAUSS3, PARTOTWO, PARTODC, DCTOTHREE,
     AND PHI DECKS;
COMMENT: NOTATION CORRESPONDENCE TABLE BETWEEN SECTION 6 (LEFT) AND
     PROGRAM (RIGHT): D1=D. D2=D2. D3=D3. E1=E14,E25. E2=E1=E11. E2=E33.
     E3=E21. E4=F22. E5=E23. E6=E41. E7=E43. E8=E51. E9=E52. E10=E53.
     E11=E62. E12=E63. E13=E71,-E74. E14=E72. E15=E73. E16=E81. E17=E83.
     E18=E91. E19=E92,-E95. E20=E93. N0=NA N1=NB. N2=N3. N=P. P1=Q.
     P2=R. H01=K1. H02=K2. H03=K3. H11=K4. H13=K5;
REAL W1,W2,W3,K1,K2,K3,K4,K5,P,Q,R,ALPH,NB,NA,N3,NBR,NAB,NABB,NBB,Z1,Z2,
     Z3,Z4,CA,SA,NABC,NABS,P11,P12,P21,P22,FPI,Y0,Y1,Y2,Y3,Y4,EPS,H,
     E21,E22,E23,E31,E32,E33,RAD,T1,T2,T3,T4,TIM;
INTEGER I;
BOOLEAN BOOL;
ARRAY Y[0:4],DC[0:3,0:3];
LABEL LOOP,EXIT,SING;
LIST SYSTEM(K1,K2,K3,K4,K5,P,Q,R,ALPH),
     HISTORY(P11,P12,P21,P22,T1,T2,T3,T4);

PROCEDURE KUTTAMERSON(N, X,HH, Y, F, EPS, AB , CUT, ERROR, STEPSIZE);
    VALUE N,HH,EPS,AB,CUT,STEPSIZE; INTEGER N,CUT; REAL X,HH,EPS,AB;
    REAL ARRAY Y[0]; PROCEDURE F; LABEL ERROR; BOOLEAN STEPSIZE;
COMMENT EPS AND AB ARE THE RELATIVE AND ABSOLUTE ERROR BOUNDS RESP.
STEPSIZE TRUE TO WRITE STEPSIZE WHEN CHANGED, FALSE FOR NO OUTPUT.
CUT IS THE MAX NO OF TIMES THE STEP SIZE MAY BE CUT IN SUCCESSION
WITHOUT RETURNING TO ERROR;
BEGIN COMMENT KUTTA MERSON INTEGRATES A SYSTEM OF N FIRST ORDER
                ORDINARY DIFFERENTIAL EQUATIONS. SEE L. FOX, "NUMERICAL
                SOLUTION OF ORDINARY AND PARTIAL DIFFERENTIAL
                EQUATIONS", P. 24, PERMAGON PRESS, 1962 ;
    OWN REAL HC, FINAL, H2, H3, H6, H8, ERR, TEST, T, H;
    OWN INTEGER I,CU; OWN BOOLEAN DBL; LABEL L,KM,RETURN;
    OWN REAL ARRAY Y1, Y2, F0, F1, F2[0:30];
COMMENT EXCEPT FOR HC, THE OWN VARIABLES ARE FOR SPEED ONLY;
    FORMAT MSSG("THE STEP SIZE IS NOW", R12.5," AT T=",R12.5);
    DEFINE FORI = FOR I←1 STEP 1 UNTIL N DO #,
       CONSTANTS = H2←H/2.0; H3←H/3.0; H6←H/6.0; H8←H/8.0 #;
COMMENT CHECK FOR INITIAL ENTRY AND ADJUST H IF NECESSARY ;
    H←HH;
    IF N=0 THEN BEGIN HC ← H; GO TO RETURN END;
    IF H=0 THEN GO TO RETURN; FINAL ← X+H;
    IF HC=0 THEN HC ← H;
    IF EPS≠0 AND ABS(H)>ABS(HC) THEN
       IF SIGN(H)≠SIGN(HC) THEN H ← HC ← -HC ELSE H ← HC;
    CU←CUT;
```

```
    T ← X+H;  X ← FINAL;  CONSTANTS;
COMMENT  MAIN KUTTA-MERSON STEP LOOP ;
  L:FOR T←T STEP H UNTIL FINAL DO
    BEGIN  KM: F(T-H,Y,F0);
       FORI Y1[I] ← F0[I]×H3+Y[I];  F(T-2×H3, Y1, F1);
       FORI Y1[I] ← (F0[I]+F1[I])×H6+Y[I];  F(T-2×H3, Y1, F1);
       FORI Y1[I] ← (F1[I]×3.0+F0[I])×H8+Y[I];  F(T-H2, Y1, F2);
       FORI Y1[I] ← (F2[I]×4.0-F1[I]×3.0+F0[I])×H2+Y[I];  F(T, Y1, F1);
       FORI Y2[I] ← (F2[I]×4.0+F1[I]+F0[I])×H6+Y[I];
COMMENT  DOES THE STEP SIZE H NEED TO BE CHANGED  ;
       IF EPS≠0 THEN
       BEGIN  DBL ← TRUE;
          FORI BEGIN ERR←ABS(Y1[I]-Y2[I])×0.2;  TEST←ABS(Y1[I])×EPS;
             IF ERR>TEST AND ERR>AB  THEN COMMENT HALF H;
                BEGIN  H ← H2; T←T-H2;
                   IF STEPSIZE THEN WRITE([DBL], MSSG, H,T);
                   IF (CU←CU-1)<0 THEN BEGIN X←T; GO TO ERROR END;
                   IF T+H=T THEN BEGIN X←T; GO TO ERROR END;
                   CONSTANTS;  GO TO KM;
                END;
             IF 64.0×ERR>TEST THEN  DBL ← FALSE;
          END;
          IF DBL THEN BEGIN H ← 2×H;
          IF STEPSIZE AND H≤HH THEN WRITE([DBL],MSSG,H,T);
          CU←CUT;
          CONSTANTS  END DOUBLE H;
       END;
           FORI  Y[I] ← Y2[I];
    END  KUTTA MERSON LOOP;
    IF EPS=0 THEN GO TO RETURN;
COMMENT  NOW BE SURE TO HAVE T = FINAL  ;
    HC ← H;  H ← FINAL-(T-H);
    IF ABS(H)>ABS(FINAL)×1.4551915228@-11 THEN
       BEGIN T ← FINAL; EPS ← 0;  CONSTANTS; GO TO L END;
RETURN: END  KUTTA MERSON;

PROCEDURE GAUSS3(A,B,X,BOOL); ARRAY A[0,0],B,X[0]; BOOLEAN BOOL;
BEGIN COMMENT MORE ACCURATE AND FOUR TIMES FASTER THAN GAUSS FOR 3 EQNS;
 OWN REAL A11,A12,A13,A21,A22,A23,A31,A32,A33,B1,B2,B3,C1,C2,C3,C4,C5,
    C6,DEL;
 A11←A[1,1]; A12←A[1,2]; A13←A[1,3]; A21←A[2,1]; A22←A[2,2]; A23←A[2,3];
 A31←A[3,1]; A32←A[3,2]; A33←A[3,3]; B1←B[1]; B2←B[2]; B3←B[3];
 C1←A22×A33-A23×A32;
 C2←A23×A31-A21×A33;
 C3←A21×A32-A22×A31;
 C4←A23×B3-A33×B2;
 C5←A32×B2-A22×B3;
 C6←A21×B3-A31×B2;
 DEL←A11×C1+A12×C2+A13×C3;
 BOOL←FALSE;
 IF DEL≠0 THEN BEGIN
    X[1]←(B1×C1+A12×C4+A13×C5)/DEL;
    X[2]←(B1×C2-A11×C4+A13×C6)/DEL;
    X[3]←(B1×C3-A11×C5-A12×C6)/DEL END
    ELSE IF B1=0 AND B2=0 AND B3=0 THEN X[1]←X[2]←X[3]←0
    ELSE BOOL←TRUE;
END GAUSS3;

PROCEDURE PARTOTWO(E1,E2,E3,ETA,T1,T2,T3,PRINT);
```

```
 VALUE E1, E2, E3, ETA,PRINT;
 REAL E1, E2, E3, ETA, T1,T2,T3; BOOLEAN PRINT;
 BEGIN COMMENT:
               INPUT: EULER PARAMETERS E1, E2, E3 AND ETA
               OUTPUT: TWO AXIS EULER ANGLES T1, T2, T3  ;
REAL PROCEDURE ASIN(X); VALUE X; REAL X;
ASIN←IF ABS(X)>1.0 THEN 999 ELSE IF ABS(X)=1.0 THEN X×1.570796327 ELSE
ARCTAN(X/SQRT(1.0-X×X));
FORMAT F1(//"PROCEDURE PARTOTWO"//X10,"INPUT:  EULER PARAMETERS"/
           X20,"E1   =",R16.9/
           X20,"E2   =",R16.9/X20,"E3   =",R16.9/X20,"ETA  =",R16.9//
           X20,"EULER PARAMETER TEST:  E1*2 + E2*2 + E3*2 + ETA*2 =",
            R16.9 //
          X10,"OUTPUT:  TWO   AXIS EULER ANGLES"/ X20,
           "THETA 1 =" , R16.9,"  RADIANS",R16.9, "  DEGREES"/X20,
           "THETA 2 =" , R16.9,"  RADIANS",R16.9, "  DEGREES"/X20,
           "THETA 3 =" , R16.9,"  RADIANS",R16.9, "  DEGREES"//);
    REAL PMP,PPP,RAD;
     RAD ← 180.0/3.1415926536 ;
    PMP ← ARCTAN(E3/E2);
    PPP ← ARCTAN(E1/ETA);
    T1 ← PMP + PPP ;
    T3 ← PPP - PMP ;
    T2    ← 2.0×ASIN(E2/(SQRT(2.0) × COS(PMP)));
IF PRINT THEN
WRITE(F1,E1,E2,E3,ETA,E1*2+E2*2+E3*2+ETA*2,T1,T1×RAD,T2,T2×RAD,T3,
     T3×RAD);
END PARTOTWO;

 PROCEDURE PARTODC(E1,E2,E3,ETA,A,PRINT);
    VALUE E1,E2,E3,ETA,PRINT;
    REAL E1,E2,E3,ETA ;
    BOOLEAN PRINT ;
    REAL ARRAY A[0,0] ;
    BEGIN COMMENT:
          INPUT:   EULER PARAMETERS E1,E2,E3 AND ETA
                   WHICH DESCRIBE A ROTATION FROM X TO Y
          OUTPUT: ARRAY OF DIRECTION COSINES A[0:3,0:3]
                   WHERE Y = AX  AND  A[I,J] = Y[I],X[J] ;
FORMAT F1(//"PROCEDURE PARTODC"//X10,"INPUT:  EULER PARAMETERS"/
           X20,"E1   =",R16.9/
           X20,"E2   =",R16.9/X20,"E3   =",R16.9/X20,"ETA  =",R16.9//
           X20,"EULER PARAMETER TEST:  E1*2 + E2*2 + E3*2 + ETA*2 =",
            R16.9 //
          "OUTPUT: DIRECTION COSINES"/ 3(X14,3R16.9/)/);
    INTEGER I,J;
    A[1,1] ←1.0-E2*2-E3*2; A[1,2]← E1×E2+E3×ETA; A[1,3]←E1×E3-E2×ETA ;
    A[2,1] ← E2×E1-E3×ETA; A[2,2]←1.0-E3*2-E1*2; A[2,3]←E2×E3+E1×ETA ;
    A[3,1] ← E3×E1+E2×ETA; A[3,2]← E3×E2-E1×ETA; A[3,3]←1.0-E1*2-E2*2;
    IF PRINT THEN
 WRITE(F1,E1,E2,E3,ETA,E1*2+E2*2+E3*2+ETA*2,
                      FOR I←1,2,3 DO FOR J←1,2,3 DO A[I,J] ) ;
 END PARTODC ;

PROCEDURE DCTOTHREE(A,T1,T2,T3,PRINT);REAL ARRAY A[0,0]; REAL T1,T2,T3;
    BOOLEAN PRINT;       BEGIN COMMENT:
          INPUT:  ARRAY OF DIRECTION COSINES A[0:3,0:3]
                   WHERE Y = AX  AND  A[I,J] = Y[I],X[J]
          OUTPUT:  THREE AXIS EULER ANGLES T1,T2, AND T3
```

```
                    WHICH DESCRIBE A ROTATION FROM X TO Y ;
REAL PROCEDURE ASIN(X); VALUE X; REAL X;
ASIN←IF ABS(X)>1.0 THEN 999 ELSE IF ABS(X)=1.0 THEN X×1.570796327 ELSE
ARCTAN(X/SQRT(1.0-X×X));
FORMAT F1(//"PROCEDURE DCTOTHREE"//X10,"INPUT:  DIRECTION COSINES"/
          3(X14,3R16.9/)//X18,"DIRECTION COSINE TESTS =",4R16.9//),
       F2(X10,"OUTPUT:  THREE AXIS EULER ANGLES"/ X20,
           "THETA 1 =" , R16.9,"  RADIANS",R16.9, "  DEGREES"/X20,
           "THETA 2 =" , R16.9,"  RADIANS",R16.9, "  DEGREES"/X20,
           "THETA 3 =" , R16.9,"  RADIANS",R16.9, "  DEGREES"//);
    INTEGER I,J;
   REAL ERR,S1,S2,S3,C1,C2,C3,PI,TEST1,TEST2,TEST3,TEST4;
   LABEL CHECK,RETURN,CANT;
PI ← 3.1415926536;
   ERR ← 0.001 ;
   S2 ← A[3,1] ;
   T2 ← ASIN(S2);
   C2 ← COS(T2);
   S1 ← -A[3,2]/C2;
   T1 ← ASIN(S1);
   C1 ← COS(T1);
IF C2 × C1 × A[3,3] < 0 THEN BEGIN T2 ← PI - T2; C2 ← -C2;
                                  T1 ← -T1 ; S1 ← -S1 ; END;
   S3 ← -A[2,1]/C2;
   T3 ← ASIN(S3);
   C3 ← COS(T3);
IF C2 × C3 × A[1,1] < 0 THEN BEGIN T3 ← PI - T3 ; C3 ← -C3; END;
   CHECK:
   IF ABS(C1 × S3 + S1 × S2 × C3 - A[1,2] ) < ERR AND
      ABS(C1 × C3 - S1 × S2 × S3 - A[2,2] ) < ERR AND
      ABS(S1 × S3 - C1 × S2 × C3 - A[1,3] ) < ERR AND
      ABS(S1 × C3 + C1 × S2 × S3 - A[2,3] ) < ERR THEN
   BEGIN
   T2 ← (T2 + PI) MOD (2 × PI) - PI;
   T3 ← (T3 + PI) MOD (2 × PI) - PI;
IF PRINT THEN BEGIN
TEST1←TEST2←TEST3←TEST4←0.0;
FOR I ← 1,2,3 DO BEGIN
    TEST1 ← TEST1 + A[I,1]×A[I,2];
    TEST2 ← TEST2 + A[I,1]×A[I,3];
    TEST3 ← TEST3 + A[1,I]×A[2,I];
    TEST4 ← TEST4 + A[1,I]×A[3,I];       END;
WRITE(F1,FOR I←1,2,3 DO FOR J←1,2,3 DO A[I,J],TEST1,TEST2,TEST3,TEST4);
WRITE(F2,T1,T1×180.0/PI,T2,T2×180.0/PI,T3,T3×180.0/PI);    END PRINT;
   GO TO RETURN END ELSE
    CANT: WRITE(<"THREE-AXIS EULER ANGLES CANNOT BE FOUND">);
RETURN: END DCTOTHREE;

PROCEDURE PHI(T,T1,T2,P1,P2,PER1,PER2,P,PD);
COMMENT: GENERATES INTERNAL VARIABLES ACCORDING TO THE STANDARD TIME
     HISTORY.  SEE SECTION 4.4;
COMMENT: T1,PER1/2,T2,PER2/2 ARE, RESP., THE DURATIONS OF THE 4 PORTIONS
     OF THE HISTORY.  PD IS THE DERIVATIVE OF P;
VALUE T,T1,T2,P1,P2,PER1,PER2;
REAL T,T1,T2,P1,P2,PER1,PER2,P,PD;
BEGIN OWN REAL X,S;
PER1←PER1/2;
IF T≤T1 OR T≥T1+T2+PER1+PER2/2 THEN BEGIN
   P←P1; PD←0     END ELSE
```

```
    IF T<T1+PER1 THEN BEGIN
       PER1←PER1+PER1;
       T←(T-T1)/PER1;
       X←FPI×T;
       S←SIN(X);
       PD←2×(P2-P1);
       P←PD×(T-S/FPI)+P1;
       PD←PD×(1-COS(X))/PER1 END ELSE
    IF T≤T1+T2+PER1 THEN BEGIN
    P←P2; PD←0      END ELSE BEGIN
       T←(T-T1-T2-PER1)/PER2;
       X←FPI×T;
       S←SIN(X);
       PD←2×(P1-P2);
       P←PD×(T-S/FPI)+P2;
       PD←PD×(1-COS(X))/PER2 END
END PHI;

PROCEDURE DERIV(X,Y,DY);
COMMENT: EQUATIONS OF MOTION.  EQUIVALENT TO EQNS (6.2.10)-(6.2.15);
VALUE X;
REAL X;
ARRAY Y,DY[0];
BEGIN OWN REAL P1,P1D,P2,P2D,S1,C1,S2,C2,D2,D3,D,E11,E14,E21,E22,E23,
     E25,E33,E41,E43,E51,E52,E53,E62,E63,E71,E72,E73,E74,E81,E83,E91,
     E92,E93,E95,Y1,Y2,Y3,Y4;
OWN ARRAY G[0:3,0:3],H,W[0:3];
PHI(X,T1,T3,P11,P12,1,1,P1,P1D);     PHI(X,T2,T4,P21,P22,1,1,P2,P2D);
S1←SIN(P1); C1←COS(P1);              S2←SIN(P2); C2←COS(P2);
Z1←NBB×Q×C1×C2;                      Z3←NBR×S1;
D2←-Z1+Z2+Z3-NABC×C1;                D3←-Z1+Z2-NABC×C2+NABS×S2;
D←2×(D3+Z3-NABC×C1);                 E21←-NBR×S2;
E33←2×NB×R↑2/Q;                      E52←NAB×S1;
E25←E14←-K4-NB×N3×Q;                 E11←E33-E14;
E41←2×NAB/Q;                         E22←K2+E41-Z3-E14;
E41←E41+K1;                          E43←-E41×SA;
E41←E41×CA;                          E62←-E14×C1;
E63←E14×S1;                          E72←-NBR+Z4×S1;
E73←NBB×Q×C2+NABC+Z4;                E74←NBB×Q×S2-NABS;
E71←-E74;                            E81←-E14×C2;
E83←-E14×S2;                         E91←NABS-Z4×S2;
E92←NBB×Q;                           E93←NABC+E92×C1+Z4×C2;
E95←-(E92←E92×S1+NBR);               E23←NBR×(C2-C1);
E51←-NAB×S2+K3×SA;                   E53←NAB×(C1+C2)+K3×CA;
H[1]←(D2-E14)×P1D+E95×S2×P2D;        H[2]←-E74×S1×P1D-(E25-D3)×P2D;
H[3]←-E74×C1×P1D-E95×C2×P2D;
G[1,1]←D+E11+E41×CA+E51×SA+E81×C2-E91×S2;
G[1,2]←E52×SA-E92×S2;
G[1,3]←E43×CA+E53×SA+E83×C2-E93×S2;
G[2,1]←E21+E71×S1;
G[2,2]←D+E22+E62×C1+E72×S1;
G[2,3]←E23+E63×C1+E73×S1;
G[3,1]←-E41×SA+E51×CA+E71×C1+E81×S2+E91×C2;
G[3,2]←E52×CA-E62×S1+E72×C1+E92×C2;
G[3,3]←D+E33-E43×SA+E53×CA-E63×S1+E73×C1+E83×S2+E93×C2;
GAUSS3(G,H,W,BOOL); IF BOOL THEN GO TO SING;
COMMENT: W=ANGULAR RATES, Y=EULER PARAMETERS;
W1←W[1]; W2←W[2]; W3←W[3];
Y1←Y[1]; Y2←Y[2]; Y3←Y[3]; Y4←Y[4];
```

```
DY[1]←(W3×Y2-W2×Y3+W1×Y4)/2;
DY[2]←(-W3×Y1+W1×Y3+W2×Y4)/2;
DY[3]←(W2×Y1-W1×Y2+W3×Y4)/2;
DY[4]←-(W1×Y1+W2×Y2+W3×Y3)/2
END PROCEDURE DERIV;

COMMENT: FIRST EXECUTABLE STATEMENT;
FPI←16×ARCTAN(1);
RAD←720/FPI; COMMENT: FPI=4 TIMES PI, RAD CONVERTS RADIANS-DEGREES;
LOOP: READ(SYSTEM)[EXIT];
READ(HISTORY)[EXIT];
READ(EPS,H)[EXIT]; COMMENT EPS=INTEGRATION ERROR PARAMETER, H=STEP SIZE;
WRITE(<"INTEGRATION OF THE EQUATIONS OF MOTION FOR THE THREE BODY MODEL"
     ,",  SEE SECTION 6",///,"INPUT DATA",//,"      INERTIA PARAMETERS: "
     ,"K1,...,K5 ",5R17.5,//,"      P=MA/MB, Q=C/B, R=A/B, ALPHA  ",
     4R17.5,/>,SYSTEM);
WRITE(<"      HISTORY: P11, P12, P21, P22    ",4R17.5,//,
     X14,"T1, T2, T3, T4",X7 ,4R17.5>,HISTORY);
WRITE(<///,"INTEGRATION ERROR PARAMETER",R16.4>,EPS);
P11←P11/RAD; P12←P12/RAD; P21←P21/RAD; P22←P22/RAD;
NB←1/(P+2);                          NA←P×NB;
N3←(NA+NB)*2+NB*2;                   NBR←NB×R;
NAB←NA×NB;                           NABB←NAB×NB;
NBB←NB*2;                            Z2←NABB×Q;
Z4←K5-Z2;                            CA←COS(ALPH);
SA←SIN(ALPH);                        NABC←NAB×CA;
NABS←NAB×SA;
WRITE([DBL],</// ,X10,"TIME",X13,"EPS 1",X12,"EPS 2",X12,"EPS 3",X12,
   "ETA",X14,"XI (DEG)">);
YO←Y[1]←Y[2]←Y[3]←0; Y[4]←SQRT(2);
Y1←Y[1]; Y2←Y[2]; Y3←Y[3]; Y4←Y[4];
WRITE([DBL],<6R17.5>,YO,Y1,Y2,Y3,Y4,2×ARCTAN(SQRT(Y1*2+Y2*2+Y3*2)/Y4)
     ×RAD);
TIM←TIME(2);
WHILE YO<2-@-6 DO BEGIN COMMENT INTEGRATES EQNS. FROM YO=0 TO YO=2;
     KUTTAMERSON(4,YO,H,Y,DERIV,EPS,.01×EPS,10,SING,FALSE);
     Y1←Y[1]; Y2←Y[2]; Y3←Y[3]; Y4←Y[4];
     IF Y4≠0 THEN WRITE([DBL],<6R17.5>,YO,Y1,Y2,Y3,Y4,2×ARCTAN(SQRT(Y1*2
        +Y2*2+Y3*2)/Y4)×RAD) END;
WRITE(<//"INTEGRATION TIME =",R10.3,//>,(TIME(2)-TIM)/60);
YO←SQRT(Y1*2+Y2*2+Y3*2);
PARTOTWO(Y3,Y1,Y2,Y4,E21,E22,E23,FALSE);
PARTODC(Y1,Y2,Y3,Y4,DC,FALSE);
DCTOTHREE(DC,E31,E32,E33,FALSE);
WRITE([DBL],</,"OTHER DESCRIPTIONS OF THE REORIENTATION",/,
     "DIRECTION COSINES OF THE EULER THEOREM AXIS",X8,3R17.5,/,
     "TWO-AXIS EULER ANGLES (3-1-3, DEGREES)",X13,3R17.5,/,
     "THREE-AXIS EULER ANGLES (1-2-3, DEGREES)",X11,3R17.5,/>,
     Y1/YO,Y2/YO,Y3/YO,E21×RAD,E22×RAD,E23×RAD,E31×RAD,E32×RAD,E33×RAD);
WRITE([PAGE]);
GO TO LOOP;
SING: WRITE(<"ERROR IN GAUSS3">);
WRITE([PAGE]);
GO TO LOOP;
EXIT: END.

1.287 1.256 .0641 .0188 .00047 15.6 .584 .446 0 SYSTEM
0 180 0 30                    0 .5 .5 .5          HISTORY
.01 .5                                            CONTROL
```

APPENDIX 4

The computer program (written in Extended Algol for the Burroughs B5500) used with Section 8, together with the data used to obtain Fig. 8.4.

```
BEGIN COMMENT PROGRAM USED WITH SECTION 8;
COMMENT NEEDS KUTTAMERSON, GAUSS3, DYDT, GAUSS, PLOT PACKAGE (123);
COMMENT FOR ANGLES, INPUT AND PRINTED OUTPUT IN DEGREES, PUNCHED OUTPUT
    IN RADIANS;
 REAL A3,B3,C3,A,B,C,AA,MA,MB,MC,K1,K2,K3,L1,L2,L3,M1,M2,M,N1,N2,N3,N4,
    N5,N6,RAD,PHO,PH1,PSO,PS1,XIO,XI1,EPS,NORM,T,TIM,SUM,CONST,TEMP,
    FACTOR,DEL1,DEL2,DEL3,Y4,Y5,Y6,F10,F20,SAVEEPS,CHANGE,S1,S2,C1,C2,
    PHD,PSD,XID,H1,H2,D,DH1DPH,DH1DPS,DH1DPHDPH,DH1DPSDPS,DH1DPHDPS,
    DH2DPH,DH2DPS,DH2DPHDPH,DH2DPSDPS,DH2DPHDPS,DDCPH,DDDPS,DDDPHDPH,
    DDDPSDPS,DDDPHDPS,F1,F2,DF1DPH,DF1DPS,DF1DPSDPS,DF1DPHDPS,DF2DPHDPH,
    DF2DPHDPS,G,DGDPH,DGDPS,DF2DPH,DF2DPS;
 INTEGER  I,J,K,NSTEP,COUNT,MSTEP;
 BOOLEAN PRINT,PLOT,PUNCH,MSSG,BOOL,FIRST;
 LABEL LOOP,EXIT,SING,SING1,SING2;
 LIST SYSTEM(K1,K2,K3,M1,M2,L1,L2,L3),
      BOUNDARY(PHO,PH1,PSO,PS1,XIO,XI1);
```

Insert Gauss3 and Kuttamerson Procedures here. See Appendix 3.

```
 PROCEDURE GAUSS (M, A, B, X, SING);
 COMMENT SOLVES LINEAR SYSTEMS OF EQUATIONS;
 VALUE M;
 INTEGER M;
 REAL ARRAY A [0, 0], B, X [0];
 LABEL SING;
 BEGIN INTEGER I, J, IMAX, K, L;
    REAL MX, T, QUOT;
    PROCEDURE ELIM (KP1, N, QUOT, A, B);
    VALUE KP1, N, QUOT;
    INTEGER N, KP1;
    REAL QUOT;
    REAL ARRAY A, B [0];
    BEGIN OWN INTEGER I;
       FOR I ← KP1 STEP 1 UNTIL N DO A [I] ← A [I] - QUOT × B [I];
    END ELIM;
    FOR K ← 1 STEP 1 UNTIL M DO
    BEGIN MX ← 0;
       FOR I ← K STEP 1 UNTIL M DO IF MX < (T←ABS(A[I,K])) THEN
       BEGIN MX ← T; IMAX←I END;
       IF MX = 0 THEN GO TO SING;
       IF K ≠ IMAX THEN
       BEGIN
          T ← B [K];
          B [K] ← B [IMAX];
```

```
            B [IMAX]←T;
            FOR L ←M STEP -1 UNTIL K DO
            BEGIN T ← A [IMAX,L];
               A [IMAX,L]←A[K,L];
               A [K, L] ← T;
            END;
         END ELSE T←A[K,K];
         FOR J ← K + 1 STEP 1 UNTIL M DO
         BEGIN QUOT ← A [J, K] / T;
            B [J] ← B [J] - B [K] × QUOT;
            ELIM (K + 1, M, QUOT, A [J, *], A [K, *]);
         END;
      END;
      FOR I ← M STEP - 1 UNTIL 1 DO
      BEGIN T ← 0;
         FOR L ← I + 1 STEP 1 UNTIL M DO T ← T + A [I, L] × X [L];
         X [I] ← (B [I] - T) / A [I, I];
      END;
   END OF GAUSS;

PROCEDURE GRAPHPLOTTER(X,Y,M,N,G,L,S,EM,C0,C1,C2,C3,C4);VALUE M,N,G,L,S;
INTEGER M,N,G,L,S;ALPHA EM,C0,C1,C2,C3,C4;REAL ARRAY X[0],Y[0,0];BEGIN M
←M-1;N←N-1;G←G-1;L←L-1;BEGIN REAL P,Q,XMAX,XMIN,YMAX,YMIN;INTEGER I,J;AL
PHA ARRAY PLOT[0:L,0:S+G],C[0:3];C[0]←C1;C[1]←C2;C[2]←C3;C[3]←C4;XMAX←XM
IN←X[0];FOR I←1STEP 1UNTIL M DO BEGIN IF X[I]>XMAX THEN XMAX←X[I];IF X[I
]<XMIN THEN XMIN←X[I]END MAXMIN HUNT FOR X;YMAX←YMIN←Y[0,0];FOR I←0STEP
1UNTIL M DO FOR J←0STEP 1UNTIL N DO BEGIN IF Y[I,J]>YMAX THEN YMAX←Y[I,J
];IF Y[I,J]<YMIN THEN YMIN←Y[I,J]END MAXMIN HUNT FOR Y;P←L/(XMAX-XMIN);Q
←(S-1)/(YMAX-YMIN);FOR I←0STEP 1UNTIL L DO FOR J←0STEP 1UNTIL G DO PLOT[
I,J]←"      ";S←S+G;G←G+1;FOR I←0STEP 1UNTIL L DO FOR J←G STEP 1UNTIL S
DO PLOT[I,J]←C0;FOR I←0,L DO FOR J←G STEP 1UNTIL S DO PLOT[I,J]←EM;FOR I
←1STEP 1UNTIL L-1DO FOR J←G,S DO PLOT[I,J]←EM;FOR I←0STEP 1UNTIL M DO FO
R J←N STEP-1UNTIL 0DO PLOT[ENTIER(0.5+P×(X[I]-XMIN)),G+ENTIER(0.5+Q×(Y[I
,J]-YMIN))]←C[J];FOR I←0STEP 1UNTIL L DO WRITE(<12CA1>,FOR J←0STEP 1UNTI
L S DO PLOT[I,J]);END END OF GRAPHPLOTTER WITH NO XMAX  ETC;

PROCEDURE CORRECTION(M,N,Y,PRO,C1,C2);
   VALUE M,N; INTEGER M,N; ARRAY Y,C1,C2[0]; PROCEDURE PRO;
COMMENT M+1= NUMBER OF POINTS IN Y: Y[0:M],
   N= HIGHEST DIFFERENCES TO BE USED: N= 3,...,10,
   Y= ARRAY FOR WHICH DIFFERENCE CORRECTIONS WILL BE COMPUTED,
   PRO(YY,M,K)= PROCEDURE WHICH WILL EXTRAPOLATE YY BY COMPUTING YY[-K]
      & YY[M+K] GIVEN YY[-K+1],...,YY[M+K-1],
   C1,C2= THE DIFFERENCE CORRECTIONS FOR THE FIRST AND SECOND
      DERIVATIVES OF Y.  SEE REF.[46];
BEGIN INTEGER I,N2,J;
 ARRAY EVEN,ODD[-ENTIER((N-1)/2):M+ENTIER((N-1)/2)], A[0:10];
 N2← ENTIER((N-1)/2);
 FOR I←0 STEP 1 UNTIL M DO EVEN[I]←Y[I];
 FOR I←1 STEP 1 UNTIL N2 DO PRO(EVEN,M,I);
 FOR I←1 STEP 1 UNTIL M-1 DO C1[I]←C2[I]←0;
 A[3]←-1/6; A[4]←-1/12; A[5]←1/30; A[6]←1/90; A[7]←-1/140; A[8]←-1/560;
    A[9]←1/630; A[10]←1/3150;
 FOR J←1 STEP 1 UNTIL N DO IF J MOD 2≠ 0 THEN
    FOR I←-N2+ENTIER(J/2) STEP 1 UNTIL M+N2-ENTIER((J+1)/2) DO BEGIN
       ODD[I]←EVEN[I+1]-EVEN[I];
       IF J>2 AND I>0 AND I<M THEN C1[I]←C1[I]+A[J]/2×(ODD[I-1]+ODD[I])
    END ELSE
    FOR I←-N2+ENTIER(J/2) STEP 1 UNTIL M+N2-ENTIER((J+1)/2) DO BEGIN
```

```
        EVEN[I]←ODD[I]-ODD[I-1];
        IF J>2 AND I>0 AND I<M THEN C2[I]←C2[I]+A[J]×EVEN[I] END;
    END CORRECTION;

REAL PROCEDURE DYDT(Y,K,NSTEP);
    VALUE K,NSTEP; INTEGER K,NSTEP; ARRAY Y[0];
    COMMENT THE FIRST DERIVATIVE OF Y AT Y[K].  SEE REF.[47];
    COMMENT NSTEP IS THE NUMBER OF INTERVALS IN A UNIT DISTANCE: NSTEP=1/H;
    DYDT← IF K≥4 AND K≤NSTEP-4 THEN NSTEP/40320.0×(144.0×(Y[K-4]-Y[K+4])
       -1536.0×(Y[K-3]-Y[K+3])+8064.0×(Y[K-2]-Y[K+2])-32256.0×(Y[K-1]
       -Y[K+1]))
       ELSE IF K≥3 AND K≤NSTEP-3 THEN NSTEP/720.0×(-12.0×(Y[K-3]-Y[K+3])
       +108.0×(Y[K-2]-Y[K+2])-540.0×(Y[K-1]-Y[K+1]))
       ELSE IF K=0 THEN NSTEP/40320.0×(-109584.0×Y[0]+322560.0×Y[1]
       -564480.0×Y[2]+752640.0×Y[3]-705600.0×Y[4]+451584.0×Y[5]
       -188160.0×Y[6]+46080.0×Y[7]-5040.0×Y[8])
       ELSE IF K=NSTEP THEN NSTEP/40320.0×(5040.0×Y[K-8]-46080.0×Y[K-7]
       +188160.0×Y[K-6]-451584.0×Y[K-5]+705600.0×Y[K-4]-752640.0×Y[K-3]
       +564480.0×Y[K-2]-322560.0×Y[K-1]+109584.0×Y[K])
       ELSE IF K<3 THEN NSTEP/720.0×(-1764.0×Y[K]+4320.0×Y[K+1]-5400.0
       ×Y[K+2]+4800.0×Y[K+3]-2700.0×Y[K+4]+864.0×Y[K+5]-120.0×Y[K+6])
       ELSE NSTEP/720.0×(120.0×Y[K-6]-864.0×Y[K-5]+2700.0×Y[K-4]
       -4800.0×Y[K-3]+5400.0×Y[K-2]-4320.0×Y[K-1]+1764.0×Y[K]);

PROCEDURE F1ANDF2;
    BEGIN COMMENT COMPUTES F1 AND F2.  SEE EQN (4.2.8);
    DH1DPS← N1×(S1×C2+C1×S2);
    DH2DPH← -DH1DPS;
    DH1DPSDPS← N1×(C1×C2-S1×S2);
    DH1DPHDPS← DH1DPSDPS;
    DH2DPHDPH← DH2DPHDPS←-DH1DPSDPS;
    DH1DPH← DH1DPS-N2×S1;
    DH1DPHDPH← DH1DPSDPS-N2×C1;
    H1← -DH1DPHDPH+N4;
    DH2DPS← -DH1DPS+N3×S2;
    DH2DPSDPS← -DH1DPSDPS+N3×C2;
    H2← -DH2DPSDPS+N5;
    DDDPH← -2.0×DH1DPH;
    DDDPS← 2.0×DH2DPS;
    DDDPHDPH← -2.0×DH1DPHDPH;
    DDDPSDPS← 2.0×DH2DPSDPS;
    DDDPHDPS← -2.0×DH1DPSDPS;
    D← -DDDPHDPH-DDDPSDPS+DDDPHDPS+N6;
    F1← H1/D;
    F2← H2/D
    END F1ANDF2;

PROCEDURE DERIVATIVES;
    BEGIN COMMENT COMPUTES PARTIAL DERIVATIVES OF F1, F2, AND G.  SEE EQNS
          (4.2.8),(8.1.8);
    DF1DPH← (DH1DPH-F1×DDDPH)/D;
    DF1DPS← (DH1DPS-F1×DDDPS)/D;
    DF2DPH← (DH2DPH-F2×DDDPH)/D;
    DF2DPS← (DH2DPS-F2×DDDPS)/D;
    DF1DPSDPS← (DH1DPSDPS-2.0×DF1DPS×DDDPS-F1×DDDPSDPS)/D;
    DF1DPHDPS← (DH1DPHDPS-DF1DPS×DDDPH-DF1DPH×DDDPS-F1×DDDPHDPS)/D;
    DF2DPHDPH← (DH2DPHDPH-2.0×DF2DPH×DDDPH-F2×DDDPHDPH)/D;
    DF2DPHDPS← (DH2DPHDPS-DF2DPS×DDDPH-DF2DPH×DDDPS-F2×DDDPHDPS)/D;
    G← DF1DPS-DF2DPH;
```

```
 DGDPH← DF1DPHDPS-DF2DPHDPH;
 DGDPS← DF1DPSDPS-DF2DPHDPS
 END DERIVATIVES;

COMMENT FIRST EXECUTABLE STATEMENT;
 RAD← 45/ARCTAN(1);
LOOP: READ(SYSTEM)[EXIT];
       READ(BOUNDARY)[EXIT];
       READ(EPS,CHANGE,NORM,COUNT,NSTEP,MSSG,PRINT,PLOT,PUNCH)[EXIT];
 WRITE(<"THREE BODY MODEL TREATED AS A SECOND PROBLEM.  SEE SECTION 8.",
    ///,"INPUT DATA",//,"     INERTIA PARAMETERS: K1, K2, K3",3R17.5,
    //,"     M1=MB/M, M2=MC/M, L1, L2, L3  ",5R17.5,/>,SYSTEM);
 WRITE(<"     INITIAL VALUES PHI, PSI, XI   ",3R17.5,//,"     FINAL VAL"
    , "UES   PHI, PSI, XI   ",3R17.5>,PH0,PS0,XI0,PH1,PS1,XI1);
 WRITE(<///,"EPSILON, CHANGE, NORM ERROR, NSTEP ",
    4R17.5,///>,EPS,CHANGE,NORM,NSTEP);
 EPS←EPS×10; COMMENT EPS IS RESTORED TO ITS INITIAL VALUE AFTER THE
    FINITE DIFFERENCE SOLN;
 MSTEP←8; COMMENT MSTEP=N IN SECTION 8;
COMMENT CONVERT BOUNDARY VALUES TO RADIANS;
 PH0← PH0/RAD; PH1← PH1/RAD;
 PS0← PS0/RAD; PS1← PS1/RAD;
 XI0← XI0/RAD; XI1← XI1/RAD;
COMMENT CORRESPONDENCE OF SECTION 8 (LEFT) TO PROGRAM (RIGHT): I0=A3.
    I1=B3. I2=C3. M0=MA. M1=MB. M2=MC. A0=A. A1=B. A2=C. B0=AA;
 A3← K1; B3← K2; C3← K3;
 MA← 1-M1-M2; MB← M1; MC← M2;
 A← L1; B← L2; C← L3; AA← 1;
 M← MA+MB+MC;
COMMENT N1,...,N6 ARE USED IN F1ANDF2;
 N1← MB×MC×M×B×C;
 N2← MB×M×B×(MA×A+MC×AA);
 N3← MC×M×C×((MA+MB)×AA-MA×A);
 N4← -M*2×B3-MB×M×(MA+MC)×B*2;
 N5← M*2×C3+MC×M×(MA+MB)×C*2;
 N6← M*2×(A3+B3+C3)+2.0×MB×MC×M×B×C+MA×(MB×B+MC×C)*2+MA×((MB+MC)×A-MC×AA
    )*2+MB×((MA+MC)×B-MC×C)*2+MB×(MA×A+MC×AA)*2+MC×((MA+MB)×C-MB×B)*2
    +MC×((MA+MB)×AA-MA×A)*2;

COMMENT NEW BLOCK HERE TO REDUCE STORAGE REQUIRED;
BEGIN ARRAY PH,PS,XI[0:NSTEP], CE1,CE2[0:3×(MSTEP-1)],
    KK,KKK,X[0:3×MSTEP-2],F[0:3×MSTEP-2,0:3×MSTEP-2];
 LABEL APPROX;

PROCEDURE DERIV(T,Y,DY); VALUE T; REAL T; ARRAY Y,DY[0];
COMMENT EQNS FOR THE INITIAL VALUE PROBLEM.  SEE EQNS (8.3.1)-(8.3.4);
 BEGIN OWN REAL Y1,Y2,Y3;
 OWN REAL ARRAY P[0:3,0:3],Q[0:3];
 DY[4]← Y1← Y[1]; DY[5]← Y2← Y[2]; DY[6]← Y3← Y[3];
 S1← SIN(Y[4]); C1← COS(Y[4]);
 S2← SIN(Y[5]); C2← COS(Y[5]);
 F1ANDF2;
 DERIVATIVES;
 P[1,2]← P[2,1]← 0;
 P[1,1]← P[2,2]← 1.0;
 P[1,3]← P[3,1]← F1;
 P[2,3]← P[3,2]← F2;
 P[3,3]← -1.0;
 Q[1]← -G×Y2×Y3;
```

```
 Q[2]← G×Y1×Y3;
 Q[3]← -DF1DPH×Y1*2-(DF1DPS+DF2DPH)×Y1×Y2-DF2DPS×Y2*2;
 GAUSS3(P,Q,DY,BOOL); IF BOOL THEN GO TO SING
END DERIV;

PROCEDURE DIFFEQ(T,X,DX); VALUE T; REAL T; ARRAY X,DX[0];
COMMENT EQNS FOR SOLVING THE NONLINEAR FINITE DIFF EQNS.  SEE, E.G.,
    EQN (8.2.13);
 BEGIN OWN REAL U0,U1,U2,V0,V1,V2,W0,W1,W2,DU,DV,DW,W,CE10,CE11,CE12,
    CE20,CE21,CE22,A,B,C;
 OWN INTEGER I0,I1,I2;
 FOR I← 0 STEP 1 UNTIL 3×MSTEP-2 DO FOR J← 1 STEP 1 UNTIL 3×MSTEP-2 DO
    F[I,J]← 0;
 FOR I← 1 STEP 1 UNTIL MSTEP-1 DO BEGIN
COMMENT THIS LOOP FILLS COEFFICIENTS (8.2.13) AND CORRESPONDING EQNS
    OBTAINED FROM (8.2.8),(8.2.9);
 I0←3×I; I1←I0-1; I2←I1-1;
 IF I=1 THEN BEGIN U0← PH0; V0← PS0; W0← XI0 END
    ELSE BEGIN U0← X[I2-3]; V0←X[I1-3]; W0←X[I0-3] END;
 U1← X[I2]; V1←X[I1]; W1←X[I0];
  IF I=MSTEP-1 THEN BEGIN U2← PH1; V2← PS1; W2← X[3×MSTEP-2] END ELSE
    BEGIN U2← X[I2+3]; V2←X[I1+3]; W2←X[I0+3] END;
 DU←U2-U0; DV←V2-V0; DW←W2-W0; W←W0-2.0×W1+W2;
 CE10←CE1[I0]; CE11←CE1[I1]; CE12←CE1[I2];
 CE20←CE2[I0]; CE21←CE2[I1]; CE22←CE2[I2];
 S1← SIN(U1); C1← COS(U1);
 S2← SIN(V1); C2← COS(V1);
 F1ANDF2; DERIVATIVES;
 G←G/4.0; DGDPH←DGDPH/4.0; DGDPS←DGDPS/4.0;
 A←DW+2.0×CE10;
 B←A×(DV+2.0×CE11);
 C←W+CE20;
 IF I>1 THEN BEGIN
    F[I2,I2-3]←1.0;
    F[I2,I1-3]←-G×A;
    F[I2,I0-3]←F1-G×(DV+2.0×CE11);
    F[I1,I2-3]←G×A;
    F[I1,I1-3]←1.0;
    F[I1,I0-3]←F2+G×(DU+2.0×CE12);
    F[I0,I2-3]←-F1;
    F[I0,I1-3]←-F2;
    F[I0,I0-3]←1.0 END;
 F[I2,I2]←-2.0+DF1DPH×C+DGDPH×B;
 F[I2,I1]←DF1DPS×C+DGDPS×B;
 F[I2,I0]←-2.0×F1;
 B←A×(DU+2.0×CE12);
 F[I1,I2]←DF2DPH×C-DGDPH×B;
 F[I1,I1]←-2.0+DF2DPS×C-DGDPS×B;
 F[I1,I0]←-2.0×F2;
 C←DU+2.0×CE12;
 B←DV+2.0×CE11;
 F[I0,I2]←DF1DPH×C+DF2DPH×B;
 F[I0,I1]←DF1DPS×C+DF2DPS×B;
 IF I<MSTEP-1 THEN BEGIN
    F[I2,I2+3]←1.0;
    F[I2,I1+3]←G×A;
    F[I2,I0+3]←F1+G×B;
    F[I1,I2+3]←-G×A;
    F[I1,I1+3]←1.0;
```

```
      F[I1,I0+3]←F2-G×C;
      F[I0,I2+3]←F1;
      F[I0,I1+3]←F2;
      F[I0,I0+3]←-1.0;
      F[I0,3×MSTEP-2]←2.0/MSTEP
      END ELSE BEGIN
      F[I2,3×MSTEP-2]←F1+G×B;
      F[I1,3×MSTEP-2]←F2-G×C;
      F[I0,3×MSTEP-2]←-1.0+2.0/MSTEP END
   END LOOP;
   COMMENT NOW FILL COEFFICIENTS OF EQN CORRESPONDING TO (8.2.11);
   S1← SIN(PH1); C1← COS(PH1);
   S2← SIN(PS1); C2← COS(PS1);
   F1ANDF2;
   I0←3×MSTEP; I2←I0-2;
   IF MSTEP>8 THEN BEGIN A←5040.0/20160.0;
      F[I2,I0-26]←A×F1;
      F[I2,I0-25]←A×F2;
      F[I2,I0-24]←-A END;
   A←-46080.0/20160.0;
   F[I2,I0-23]←A×F1;
   F[I2,I0-22]←A×F2;
   F[I2,I0-21]←-A;
   A←188160.0/20160.0;
   F[I2,I0-20]←A×F1;
   F[I2,I0-19]←A×F2;
   F[I2,I0-18]←-A;
   A←-451584.0/20160.0;
   F[I2,I0-17]←A×F1;
   F[I2,I0-16]←A×F2;
   F[I2,I0-15]←-A;
   A←705600.0/20160.0;
   F[I2,I0-14]←A×F1;
   F[I2,I0-13]←A×F2;
   F[I2,I0-12]←-A;
   A←-752640.0/20160.0;
   F[I2,I0-11]←A×F1;
   F[I2,I0-10]←A×F2;
   F[I2,I0- 9]←-A;
   A←564480.0/20160.0;
   F[I2,I0- 8]←A×F1;
   F[I2,I0- 7]←A×F2;
   F[I2,I0- 6]←-A;
   A←-322560.0/20160.0;
   F[I2,I0- 5]←A×F1;
   F[I2,I0- 4]←A×F2;
   F[I2,I0- 3]←-A;
   F[I2,I0- 2]←-109584.0/20160.0+2.0/MSTEP;
   FOR I← 1 STEP 1 UNTIL 3×MSTEP-2 DO KKK[I]←-KK[I];
   GAUSS(3×MSTEP-2,F,KKK,DX,SING1)
END DIFFEQ;

PROCEDURE EXTRAPOLATE(Y,M,K); VALUE M,K; INTEGER M,K;  ARRAY Y[*];
COMMENT EXTRAPOLATE IS CALLED BY CORRECTION;
BEGIN OWN INTEGER COUNT,I;
       REAL T,D1,D2,D3;
       OWN REAL ARRAY X[0:6],PHI,PSI,CHI[-3:M+3];
       LABEL L1,L2,L3,RETURN;
       SWITCH SW← L1,RETURN,RETURN,L2,RETURN,RETURN,L3,RETURN,RETURN;
```

```
 GO TO SW[((COUNT←COUNT+1)-1) MOD 9+1];
 L1: T←.5;
    X[1]←D1←DYDT(PH,M/2,M);
    X[2]←D2←DYDT(PS,M/2,M);
    X[3]←D3←DYDT(XI,M/2,M);
    X[4]←PH[M/2]; X[5]←PS[M/2]; X[6]←XI[M/2];
    FOR I←1 STEP 1 UNTIL M/2+3 DO BEGIN
    KUTTAMERSON(6,T, 1/M,X,DERIV,EPS,.01×EPS,8,SING,MSSG);
    PHI[M/2+I]←X[4]; PSI[M/2+I]←X[5]; CHI[M/2+I]←X[6] END;
    T←.5;
    X[4]←PH[M/2]; X[5]←PS[M/2]; X[6]←XI[M/2];
    X[1]←D1; X[2]←D2; X[3]←D3;
    FOR I←-1  STEP -1 UNTIL -M/2-3 DO BEGIN
    KUTTAMERSON(6,T,-1/M,X,DERIV,EPS,.01×EPS,8,SING,MSSG);
    PHI[M/2+I]←X[4]; PSI[M/2+I]←X[5]; CHI[M/2+I]←X[6] END;
    PHI[M/2]←PH[M/2]; PSI[M/2]←PS[M/2]; CHI[M/2]←XI[M/2];
    FOR I←-3 STEP 1 UNTIL M+3 DO Y[I]←PHI[I];
    GO TO RETURN;
 L2: FOR I←-3 STEP 1 UNTIL M+3 DO Y[I]←PSI[I];
    GO TO RETURN;
 L3: FOR I←-3 STEP 1 UNTIL M+3 DO Y[I]←CHI[I];
RETURN: END EXTRAPOLATE;

 PH[0]←PH0; PH[MSTEP]←PH1;
 PS[0]←PS0; PS[MSTEP]←PS1;
 XI[0]←XI0;
FIRST←TRUE;
COMMENT INITIAL GUESS - READ FROM CARDS;
 READ(FOR I←1 STEP 1 UNTIL MSTEP-1 DO [PH[I],PS[I],XI[I]],XI[MSTEP])
    [EXIT];
COMMENT XI0,XI1 REFER TO XI IN SECTION 8, BUT XI[I] REFERS TO CHI
    (EXCEPT IN THE OUTPUT WHERE XI[I] REFERS TO XI);
 FOR I←1 STEP 1 UNTIL 3×(MSTEP-1) DO CE1[I]←CE2[I]←0;
 TIM←TIME(2);
COMMENT START OF THE FINITE DIFFERENCE SOLUTION;
APPROX:
 FOR  I← 1 STEP 1 UNTIL MSTEP-1 DO BEGIN
 S1← SIN(PH[I]); C1← COS(PH[I]);
 S2← SIN(PS[I]); C2← COS(PS[I]);
 F1ANDF2;
 G← (DH1DPS-F1×DDDPS-DH2DPH+F2×DDDPH)/D;
COMMENT KK[I] ARE THE R.H.S. OF THE DIFFERENTIAL EQNS IN DIFFEQ;
 KK[3×I-2]← PH[I-1]-2×PH[I]+PH[I+1]+F1×(XI[I-1]-2×XI[I]+XI[I+1])+G/4
    ×(PS[I+1]-PS[I-1])×(XI[I+1]-XI[I-1])+CE2[3×I-2]+F1×CE2[3×I]
    +G/2×((PS[I+1]-PS[I-1])×CE1[3×I]+(XI[I+1]-XI[I-1])×CE1[3×I-1])
    +G×CE1[3×I-1]×CE1[3×I];
 KK[3×I-1]← PS[I-1]-2×PS[I]+PS[I+1]+F2×(XI[I-1]-2×XI[I]+XI[I+1])-G/4
    ×(PH[I+1]-PH[I-1])×(XI[I+1]-XI[I-1])+CE2[3×I-1]+F2×CE2[3×I]
    -G/2×((PH[I+1]-PH[I-1])×CE1[3×I]+(XI[I+1]-XI[I-1])×CE1[3×I-2])
    -G×CE1[3×I-2]×CE1[3×I];
 KK[3×I]← F1×(PH[I+1]-PH[I-1])+F2×(PS[I+1]-PS[I-1])-XI[I+1]+XI[I-1]+2/
    MSTEP×(XI[MSTEP]-XI1)+2×F1×CE1[3×I-2]+2×F2×CE1[3×I-1]-2×CE1[3×I]END;
 I← MSTEP;
 S1← SIN(PH[I]); C1← COS(PH[I]);
 S2← SIN(PS[I]); C2← COS(PS[I]);
 F1ANDF2;
 KK[3×I-2]←(F1×(5040×PH[I-8]-46080×PH[I-7]+188160×PH[I-6]-451584×PH[I-5]
    +705600×PH[I-4]-752640×PH[I-3]+564480×PH[I-2]-322560×PH[I-1]
    +109584×PH[I])+F2×(5040×PS[I-8]-46080×PS[I-7]+188160×PS[I-6]
```

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    -451584×PS[I-5]+705600×PS[I-4]-752640×PS[I-3]
    +564480×PS[I-2]-322560×PS[I-1]+109584×PS[I])-5040×XI[I-8]
    +46080×XI[I-7]-188160×XI[I-6]+451584×XI[I-5]-705600×XI[I-4]
    +752640×XI[I-3]-564480×XI[I-2]+322560×XI[I-1]+(40320/I-109584)×XI[I]
    -40320/I×XI1)/20160;
 T← 0;
 FOR I← 1 STEP 1 UNTIL MSTEP-1 DO BEGIN
    X[3×I-2]← PH[I];
    X[3×I-1]← PS[I];
    X[3×I]← XI[I] END;
 X[3×MSTEP-2]← XI[MSTEP];
 KUTTAMERSON(0          ,T,1,X,DIFFEQ,EPS,.01×EPS,3,SING2,MSSG);
 KUTTAMERSON(3×MSTEP-2,T,1,X,DIFFEQ,IF FIRST THEN 10×EPS ELSE EPS,
    .01×EPS,COUNT,SING2,MSSG);
 CONST← XI1-X[3×MSTEP-2];
COMMENT COMPUTE THE AVERAGE RELATIVE CHANGE IN THE SOLUTION;
 TEMP←T←0;
 FOR I←1 STEP 1 UNTIL MSTEP-1  DO TEMP←TEMP+ABS(X[3×I-2]-PH[I])
    +ABS(X[3×I-1]-PS[I])+ABS(X[3×I]-XI[I]);
 TEMP←TEMP+ABS(X[3×MSTEP-2]-XI[MSTEP]);
 FOR I←1 STEP 1 UNTIL 3×MSTEP-2 DO T←T+ABS(X[I]);
 SUM←TEMP/T;
 FOR I← 1 STEP 1 UNTIL MSTEP-1 DO BEGIN
    PH[I]← X[3×I-2];
    PS[I]← X[3×I-1];
    XI[I]← X[3×I] END;
 XI[MSTEP]← X[3×MSTEP-2];
COMMENT OUTPUT THE FINITE DIFFERENCE SOLN;
 WRITE([DBL],<"APPROX SOLN: CONST=",R12.5,"   AVG REL CHANGE=",R8.1,
    "   TIME REQD=",F6.2   >,CONST,SUM,(-TIM+(TIM+TIME(2)))/60);
 IF SUM > CHANGE OR FIRST THEN BEGIN
 COMMENT COMPUTE FINITE DIFFERENCE CORRECTIONS CE1,CE2;
 CORRECTION(MSTEP,8,PH,EXTRAPOLATE,KK,KKK);
 FOR I←1 STEP 1 UNTIL MSTEP-1 DO BEGIN
    CE1[3×I-2]←KK[I];
    CE2[3×I-2]←KKK[I] END;
 CORRECTION(MSTEP,8,PS,EXTRAPOLATE,KK,KKK);
 FOR I←1 STEP 1 UNTIL MSTEP-1 DO BEGIN
    CE1[3×I-1]←KK[I];
    CE2[3×I-1]←KKK[I] END;
 CORRECTION(MSTEP,8,XI,EXTRAPOLATE,KK,KKK);
 FOR I←1 STEP 1 UNTIL MSTEP-1 DO BEGIN
    CE1[3×I  ]←KK[I];
    CE2[3×I  ]←KKK[I] END;
 FIRST←FALSE;
 GO TO APPROX END COMPUTATION OF FINITE DIFFERENCE CORRECTIONS;

BEGIN ARRAY DD[0:3,0:3], DX,DPRIME[0:3], Y[0:6];
 LABEL INTEGRATE,FINISH;
COMMENT IMPROVEMENT OF THE FINITE DIFFERENCE SOLUTION.  SEE SECTION 8.3;
COMMENT PHI=Y[4], PSI=Y[5], CHI=Y[6];
 EPS←SAVEEPS←EPS/10;
 S1← SIN(PHO); C1← COS(PHO);
 S2← SIN(PSO); C2← COS(PSO);
 F1ANDF2;
 F10← F1; F20← F2;
COMMENT COMPUTATION OF THE INITIAL DERIVATIVES FROM SECOND ORDER
    FORMULAE;
 PHD←(PH[1]-PH[0])×MSTEP;
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 PSD←(PS[1]-PS[0])×MSTEP;
 XID←(XI[1]-XI[0])×MSTEP;
 DEL1←DYDT(PH,0,MSTEP)-PHD;
 DEL2←DYDT(PS,0,MSTEP)-PSD;
 DEL3←DYDT(XI,0,MSTEP)-XID;
 DEL1←DEL1/100; DEL2←DEL2/100; DEL3←DEL3/100;
 COMMENT DEL1, DEL2, DEL3 ARE THE INCREMENTS IN THE INITIAL DERIVATIVES;
INTEGRATE: T← 0;
COMMENT SOLUTION OF THE DIFFERENTIAL EQNS AS AN INITIAL VALUE PROBLEM;
 Y[1]← PHD;          Y[2]← PSD;          Y[3]← XID;
 Y[4]← PH0; Y[5]← PS0; Y[6]← XI0;
 KUTTAMERSON(6,T,1,Y,DERIV,EPS,.01×EPS,8,SING,MSSG);
 Y4← Y[4]; Y5← Y[5]; Y6← Y[6];
 DX[1]← Y4-PH1; DX[2]← Y5-PS1;
 DX[3]← Y6-XI1←(CONST← F10×PHD+F20×PSD-XID);
 SUM← 2×(ABS(DX[1])+ABS(DX[2])+ABS(DX[3]))/(ABS(PHC)+ABS(PH1)+ABS(PS0)
    +ABS(PS1)+ABS(XI0)+ABS(XI1));
 WRITE([DBL],<"NORM=",R8.1,"   DEL1,DEL2,DEL3=",R8.1,", ",R8.1,", ",
    R8.1,"   CONST=",R12.5,"   ITERATION TIME=",F6.2>,SUM,DEL1,DEL2,
    DEL3,CONST,(-TIM←(TIM←TIME(2)))/60);
 IF SUM<NORM THEN
    IF EPS≠SAVEEPS THEN GO TO FINISH ELSE BEGIN EPS← EPS/10;
       WRITE([DBL],<"EPSILON←EPSILON/10">);
       GO TO INTEGRATE END;
 IF (COUNT← COUNT-1)<0 THEN BEGIN
    WRITE([DBL],<"SOLN FAILED TO ATTAIN PRESCRIBED ACCURACY IN GIVEN NO.
 OF ITERATIONS.  CURRENT VALUES FOLLOW.">);
    GO TO FINISH END;
COMMENT COMPUTATION OF THE INFLUENCE MATRIX DD.  SEE (8.3.6);
 T← 0;
 IF DEL1=0 THEN DEL1←(DEL1+DEL2+DEL3)/3;
 Y[1]← PHD+DEL1;   Y[2]← PSD;          Y[3]← XID;
 Y[4]← PH0; Y[5]← PS0; Y[6]← XI0;
 KUTTAMERSON(6,T,1,Y,DERIV,EPS,.01×EPS,8,SING,MSSG);
 DD[1,1]←(Y[4]-Y4)/DEL1;
 DD[2,1]←(Y[5]-Y5)/DEL1;
 DD[3,1]←(Y[6]-Y6)/DEL1+F10;
 T← 0;
 IF DEL2=0 THEN DEL2←(DEL1+DEL2+DEL3)/3;
 Y[1]← PHD;          Y[2]← PSD+DEL2;      Y[3]← XID;
 Y[4]← PH0; Y[5]← PS0; Y[6]← XI0;
 KUTTAMERSON(6,T,1,Y,DERIV,EPS,.01×EPS,8,SING,MSSG);
 DD[1,2]←(Y[4]-Y4)/DEL2;
 DD[2,2]←(Y[5]-Y5)/DEL2;
 DD[3,2]←(Y[6]-Y6)/DEL2+F20;
 T← 0;
 IF DEL3=0 THEN DEL3←(DEL1+DEL2+DEL3)/3;
 Y[1]← PHD;          Y[2]← PSD;          Y[3]← XID+DEL3;
 Y[4]← PH0; Y[5]← PS0; Y[6]← XI0;
 KUTTAMERSON(6,T,1,Y,DERIV,EPS,.01×EPS,8,SING,MSSG);
 DD[1,3]←(Y[4]-Y4)/DEL3;
 DD[2,3]←(Y[5]-Y5)/DEL3;
 DD[3,3]←(Y[6]-Y6)/DEL3-1;
 GAUSS3(DD,DX,DPRIME,BOOL); IF BOOL THEN GO TO SING;
 DEL1← DPRIME[1]; DEL2← DPRIME[2]; DEL3←DPRIME[3];
COMMENT CORRECTION OF THE INITIAL DERIVATIVES;
 PHD← PHD-DEL1; PSD← PSD-DEL2; XID← XID-DEL3;
 GO TO INTEGRATE;
FINISH: T← 0;
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 Y[1]← PHD;          Y[2]← PSD;          Y[3]← XID;
 Y[4]← PHO; Y[5]← PSO; Y[6]← XIO;
 FOR I← 1 STEP 1 UNTIL NSTEP DO BEGIN
 KUTTAMERSON(6,T,1/NSTEP,Y,DERIV,EPS,.01×EPS,8,SING,MSSG);
 PH[I]← Y[4]; PS[I]← Y[5]; XI[I]← Y[6] END;
END IMPROVEMENT BLOCK;

 FOR I← 1 STEP 1 UNTIL NSTEP DO XI[I]← XI[I]+CONST×I/NSTEP;
COMMENT CONVERGENCE ACHEIVED - OUTPUT SOLUTION;
 IF PRINT THEN BEGIN
    WRITE(<///,X6,"TIME",X11,"PHI",X12,"PSI",X12,"XI",/>);
    FOR I← 0 STEP 1 UNTIL NSTEP DO
       WRITE(<F9.3,3R15.3>,I/NSTEP,PH[I]×RAD,PS[I]×RAD,XI[I]×RAD) END;
 IF PLOT THEN BEGIN COMMENT MAKE A ROUGH PLOT OF SOLN ON LINE PRINTER;
    ARRAY X[0:NSTEP],Y[0:NSTEP,0:2];
    WRITE([PAGE]);
    WRITE(<"PLOT OF THE SOLUTION.  1=PHI, 2=PSI, 3=XI.  T=0 AT TOP OF ",
       "PAGE."//>);
    FOR I← 0 STEP 1 UNTIL NSTEP DO BEGIN
       X[I]← I; Y[I,0]← PH[I]; Y[I,1]← PS[I]; Y[I,2]← XI[I] END;
    GRAPHPLOTTER(X,Y,NSTEP+1,3,1,NSTEP+1,119," "," ","1","2","3","4")
    END PLOT;
 IF PUNCH THEN BEGIN FILE OUT CARD 0 (2,10);
    FOR I← 0 STEP 2 UNTIL NSTEP DO
       IF I≠NSTEP THEN WRITE(CARD,<6E13.6,I2>,PH[I],PS[I],XI[I],PH[I+1],
          PS[I+1],XI[I+1],I/2)
       ELSE WRITE(CARD,<3E13.6,X39,I2>,PH[I],PS[I],XI[I],I/2) END PUNCH;
END BLOCK WHERE ARRAYS PH ETC ARE DECLARED;

 GO TO SING;
SING1: WRITE([DBL],<"ERROR IN GAUSS IN DIFFEQ AT T=",R13.6>,T);
 GO TO SING;
SING2: WRITE([DBL],<"ERROR IN KUTTAMERSON FOR DIFFEQ AT T=",R13.6>,T);
SING: WRITE([PAGE]);
 GO TO LOOP;
EXIT: END.

.229 .0684 .0166 .327 .114 .512 .795 .507     SYSTEM
180 180 0 0 0 5                               BOUNDARY
@-3 @-1 @-4  8 50 0 1 1 0                     CONTROL
2.6 .4 0 2.2 .8 0 1.9 1.1 0 1.5 1.5 0 1.9 1.1 0 2.2 .8 0 2.6 .4 0 0
```

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